Problem for 1996 April

Proposed by Dan Jurca

Prove that if m and n are integers, $0 \le m$, and $1 \le n$, then $(n!)^m m! \mid (mn)!$.

Remarks.

i.
: The symbol `|' means ``divides''.

ii.
: You might like to compute the quotient (which is, of course, asserted to be an integer).

iii.
: The proposer recently came across a reference to this result in a Canadian mathematics magazine (the *Crux Mathematicorum*), in which it was claimed to be ``well known''. Since it was not well known to the proposer, the proposer wanted to prove it is true.

Solution by the proposer

Proposition: $m,n \in \mathbf{Z}$, $0 \le m$, $1 \le n \Rightarrow (n!)^m m! \mid (mn)!$.

Proof.

The assertion clearly holds in the case m=0 and $1 \le n$, so assume from now on that $1 \le m$ and $1 \le n$.

Lemma 1: $n,x \in \mathbf{Z}$, $1 \le n \le x \Rightarrow x(x-1)(x-2)\ldots[x-(n-1)]=n!(x \parallel n)$.

Proof of lemma 1.

$x(x-1)(x-2)\ldots[x-(n-1)] = [\, x!/((x-n)!)]=n![\, x!/(n!(x-n)!)]=n!(x \parallel n)$,

proving lemma 1.

Remark: Hence n! divides the product of any n consecutive integers.

Lemma 2: $m,n \in \mathbf{Z}$, $1 \le m$, $1 \le n \Rightarrow (mn \parallel n)=m((mn-1) \parallel (n-1))$.

Proof of lemma 2.

$(mn \| n)=[(mn)!/(n!(mn-n)!)] = [((mn)(mn-1)!)/(n(n-1)!(mn-n)!)] = m[((mn-1)!)/((n-1)!(mn-n)!)]=m((mn-1) \| (n-1))$,

proving lemma 2.

We complete the proof by expressing (mn)!, the product of mn factors, as a product of m products, each of n consecutive integer factors, as follows:

$$\begin{aligned}
(mn)! &= (mn)\cdot(mn-1)\cdot(mn-2)\cdot \ldots \cdot(2)\cdot(1) \\
&= \prod_{k=1}^{m} \left\{ [(m+1-k)n]\cdot[(m+1-k)n-1]\cdot \ldots \cdot[(m+1-k)n-(n-1)] \right\} \\
&= \prod_{k=1}^{m} n! \binom{(m+1-k)n}{n} \text{, by lemma 1} \\
&= (n!)^m \prod_{k=1}^{m} \binom{(m+1-k)n}{n} \\
&= (n!)^m \prod_{k=1}^{m} (m+1-k) \binom{(m+1-k)n-1}{n-1} \text{, by lemma 2} \\
&= (n!)^m m! \prod_{k=1}^{m} \binom{(m+1-k)n-1}{n-1} .
\end{aligned}$$

The last line above exhibits the integral quotient, completing the proof.

Also solved by Eric Leong.