Problem for 1996 May

Proposed by Dan Jurca

According to the 1996 April issue of the Canadian mathematics journal *Crux Mathematicorum* the following problem was given as part of the selection test for the Balkan Olympic Team (No, not *those* Olympics; rather the International Mathematics Olympiad).

Can you do it?

Prove that the sequence $(\mathrm{Im}(z_n))_{n=1}^{\infty}$ of the imaginary parts of the complex numbers $z_n=(1+i)(2+i)\ldots(n+i)$ contains infinitely many positive and infinitely many negative numbers.

Solution by the proposer

We have $k+i=\sqrt{k^2+1}\times e^{i\theta_k}$ where $\theta_k=\tan^{-1}{}^1/_k$; hence $z_n=(\prod_{k=1}^{n}\sqrt{k^2+1})\times e^{i\theta_n}$ where $\theta_n=\sum_{k=1}^{n}\tan^{-1}{}^1/_k$. One easily shows $1 \le k \Rightarrow [\ 1/2k] < \tan^{-1}{}^1/_k < {}^1/_k$. Hence with $H_n=1+{}^1/_2+{}^1/_3+\ldots+{}^1/_n$, the n-th partial sum of the (divergent) harmonic series, we have ${}^1/_2H_n < \theta_n < H_n$. Thus for $k=0,1,2,3,\ldots$ $\exists\ n_k$ such that

$$1 \le n < n_k \Rightarrow 0 \le \theta_n \le k\pi; \quad k\pi < \theta_{nk} < (k+1)\pi.$$

Clearly k even $\Rightarrow 0 < \mathrm{Im}(z_{nk})$, k odd $\Rightarrow \mathrm{Im}(z_{nk}) < 0$.

Remark: The first few n_k and θ_{nk} are as follows:

$$\theta_1 = \theta_{n0} = 0.2500000000\pi$$
$$\theta_{17} = \theta_{n1} = 1.0072981352\pi$$
$$\theta_{396} = \theta_{n2} = 2.0003604476\pi$$
$$\theta_{9165} = \theta_{n3} = 3.0000203961\pi$$
$$\theta_{212082} = \theta_{n4} = 4.0000001066\pi$$

$$\theta_{4907734} = \theta_{n5} = 5.0000000129\pi$$

Also solved by Thomas Kim and Eric Leong.