Problem for 1996 June

Proposed by Dan Jurca

Find the exact value of $(\sqrt{50}+7)^{1/3}-(\sqrt{50}-7)^{1/3}$.

Solution by the proposer

We show $(\sqrt{50}+7)^{1/3}-(\sqrt{50}-7)^{1/3}=2$.

More generally, suppose $q \in \mathbf{R}$, the set of real numbers, and let

$$x=\sqrt{\left(\frac{q^3+3q}{2}\right)^2+1}+\left(\frac{q^3+3q}{2}\right).$$

Then $x \neq 0$ and one finds at once

$$y=\frac{1}{x}=\sqrt{\left(\frac{q^3+3q}{2}\right)^2+1}-\left(\frac{q^3+3q}{2}\right).$$

Observe that with $q=2$ we have $x=(\sqrt{50}+7)$ and $y=(\sqrt{50}-7)$.
Then

$$z=x^{1/3}-y^{1/3}=x^{1/3}-\frac{1}{x^{1/3}}=\frac{x^{2/3}-1}{x^{1/3}}=\frac{x^{4/3}-x^{2/3}}{x}.$$

Also

$$z^3=\frac{x^2-3x^{4/3}+3x^{2/3}-1}{x}=\frac{x^2-1}{x}-3\frac{x^{4/3}-x^{2/3}}{x}=\frac{x^2-1}{x}-3z.$$

Finally

$$\frac{x^2-1}{x}=x-\frac{1}{x}=x-y=q^3+3q.$$

Hence $z^3+3z-(q^3+3q)=0$, so $(z-q)[z^2+qz+(q^2+3)]=0$. Since $z \in \mathbf{R}$, it follows that $z=q$. The proposed problem corresponds to the case $q=2$.

Remark: Probably the most interesting cases occur when, as in the present example, $q \in \mathbf{Q}^+$, the set of positive rationals. The proposer found (in a mathematics magazine) the problem corresponding to $q=1$.

Also solved by Walter Becker, Al Franz, Teruyuki Hiyama, Thomas Kim, Eric Leong, and Brad Stoll