# Problem for 1996 July

Proposed by Eriko Hironaka (MSRI, Berkeley)
Communicated by Dan Jurca

Consider the ``half-Pascal's triangle'', the first seven rows of which appear as follows.

1

1 0

1 1 0

1 2 0 0

1 3 2 0 0

1 4 5 0 0 0

1 5 9 5 0 0 0

Show that the sum of the entries in the n-th row equals $(n \| (\lfloor n/2 \rfloor))$.

(The top row corresponds to n=0.)

Precisely, we define the array x of integers as follows:

$$x_{i,j}=\begin{cases} 1 & \text{if } 0 \le i \text{ and } j=0; \\ x_{i-1,j-1}+x_{i-1,j} & \text{if } 1 \le i \text{ and } 1 \le j \le \lfloor i/2 \rfloor; \\ 0 & \text{if } 1 \le i \text{ and } \lfloor i/2 \rfloor < j \le i. \end{cases}$$

Then:

$$0 \le i \Rightarrow \sum_{j=0}^{i} x_{i,j}=\binom{i}{\lfloor i/2 \rfloor} .$$

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Solution by Dan Jurca

First we define the array y for $0 \le i$ and $0 \le j \le i$ as follows.

$$y_{i,j}=\begin{cases} 1 & \text{if } 0 \le i \text{ and } j=0; \\ \dfrac{i-2j+1}{j}\dbinom{i}{j-1} & \text{if } 1 \le i \text{ and } 1 \le j \le \lfloor i/2 \rfloor; \\ 0 & \text{if } 1 \le i \text{ and } \lfloor i/2 \rfloor < j \le i. \end{cases}$$

Then, obviously, $x_{i,j}=y_{i,j}$ if either $0 \le i$ and $j=0$, or if $1 \le i$ and $\lfloor i/2 \rfloor < j \le i$. Next, we show that if $1 \le i$ and $1 \le j \le \lfloor i/2 \rfloor$, then $x_{i,j}=y_{i,j}$ as well. One easily checks that $x_{1,0}=y_{1,0}$ and $x_{1,1}=y_{1,1}$; so suppose that $2 \le i$. Then if $j=1$, we find $y_{i,j}=y_{i,1}=(i-1)(i \,||\, 0)=i-1=1+(i-2)=y_{i-1,0}+y_{i-1,1}=y_{i-1,j-1}+y_{i-1,j}$. Next, if $2 \le j \le \lfloor i/2 \rfloor$, then $1 \le j-1 \le \lfloor i/2 \rfloor -1 \le \lfloor (i-1)/2 \rfloor$; hence

$$
\begin{aligned}
y_{i,j} &= \frac{i-2j+1}{j}\binom{i}{j-1} \\
&= \frac{i-2j+1}{j}\,\frac{i!}{(j-1)!(i-j+1)!} \\
&= \frac{i-2j+1}{j}\,\frac{i}{i-j+1}\,\frac{(i-1)!}{(j-1)!(i-j)!} \\
&= \frac{i(i-2j+1)}{(i-j+1)j}\binom{i-1}{j-1} \\
&= \frac{i^2-2ij+i}{(i-j+1)j}\binom{i-1}{j-1} \\
&= \frac{ij-2j^2+2j \;+\; i^2-2ij-ij+2j^2+i-2j}{(i-j+1)j}\binom{i-1}{j-1} \\
&= \frac{(i-2j+2)j \;+\; (i-j+1)(i-2j)}{(i-j+1)j}\binom{i-1}{j-1} \\
&= \left[\frac{i-2j+2}{i-j+1} + \frac{i-2j}{j}\right]\binom{i-1}{j-1} \\
&= \frac{i-2j+2}{i-j+1}\,\frac{(i-1)!}{(j-1)!(i-j)!} + \frac{i-2j}{j}\binom{i-1}{j-1} \\
&= \frac{i-2j+2}{j-1}\,\frac{(i-1)!}{(j-2)!(i-j+1)!} + \frac{i-2j}{j}\binom{i-1}{j-1} \\
&= \frac{i-2j+2}{j-1}\binom{i-1}{j-2} + \frac{i-2j}{j}\binom{i-1}{j-1} \\
&= y_{i-1,j-1}+y_{i-1,j}.
\end{aligned}
$$

It follows that for $0 \le i$ and $0 \le j \le i$ we have $x_{i,j}=y_{i,j}$.

Thus

$$
\begin{aligned}
0 \le i \text{ and } 1 \le j \le \lfloor i/2 \rfloor \Rightarrow x_{i,j} &= \frac{i-2j+1}{j}\binom{i}{j-1} \\
&= -2\binom{i}{j-1} + \frac{i+1}{j}\binom{i}{j} \\
&= -2\binom{i}{j-1} + \binom{i+1}{j}.
\end{aligned}
$$

The following identities follow at once from the well-known $0 \le i \Rightarrow \sum_{j=0}^{i} (i \,\|\, j) = 2^i$ and the symmetry of Pascal's triangle.

$$i \text{ even} \Rightarrow \sum_{j=0}^{i/2-1} \binom{i}{j} = 2^{i-1} - \frac{1}{2}\binom{i}{i/2}$$

$$\sum_{j=0}^{i/2} \binom{i+1}{j} = 2^i$$

$$i \text{ odd} \Rightarrow \sum_{j=0}^{\lfloor i/2 \rfloor - 1} \binom{i}{j} = 2^{i-1} - \binom{i}{\lfloor i/2 \rfloor}$$

$$\sum_{j=0}^{\lfloor i/2 \rfloor} \binom{i+1}{j} = 2^i - \frac{1}{2}\binom{i+1}{(i+1)/2}$$

$$= 2^i - \binom{i}{\lfloor i/2 \rfloor} \,.$$

Hence for even i :

$$\sum_{j=0}^{i} x_{i,j} = \sum_{j=0}^{i/2} x_{i,j}$$

$$= 1 + \sum_{j=1}^{i/2} x_{i,j}$$

$$= 1 + \sum_{j=1}^{i/2} \left[ -2\binom{i}{j-1} + \binom{i+1}{j} \right]$$

$$= -2 \sum^{i/2-1} \binom{i}{j} + \sum^{i/2} \binom{i+1}{j}$$

$$\qquad j=0 \qquad\qquad j=0$$

$$=-2\left[\,2^{i-1}-\frac{1}{2}\binom{i}{i/2}\,\right]+2^{i}$$

$$=\binom{i}{i/2}\ ;$$

and for odd i :

$$\sum_{j=0}^{i} x_{i,j} = \sum_{j=0}^{\lfloor i/2\rfloor} x_{i,j}$$

$$=1+\sum_{j=1}^{\lfloor i/2\rfloor} x_{i,j}$$

$$=1+\sum_{j=1}^{\lfloor i/2\rfloor}\left[\,-2\binom{i}{j-1}+\binom{i+1}{j}\,\right]$$

$$=-2\sum_{j=0}^{\lfloor i/2\rfloor-1}\binom{i}{j}+\sum_{j=0}^{\lfloor i/2\rfloor}\binom{i+1}{j}$$

$$=-2\left[\,2^{i-1}-\binom{i}{\lfloor i/2\rfloor}\,\right]+2^{i}-\binom{i}{\lfloor i/2\rfloor}$$

$$=\binom{i}{\lfloor i/2\rfloor}$$

Thus $0 \le i \Rightarrow \sum_{j=0}^{i} x_{i,j}=(i \,\|\, (\lfloor i/2\rfloor))$, as asserted.

Also solved by the proposer (in an entirely different and quite ingenious way).