Problem for 1996 September and October

Proposed by Dan Jurca

Suppose that the rational function (quotient of polynomials) p(x)/q(x) nicely approximates the function e^x for small x, in the sense that

$$\frac{p(x)}{q(x)} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^N}{N!} + O(x^{N+1}).$$

Assume that

$$p(x) = \sum_{i=0}^{m} a_i x^i, \qquad q(x) = \sum_{i=0}^{n} b_i x^i,$$

and we define

$$\bar{p}(x) = \sum_{i=0}^{m} \bar{a}_i x^i, \qquad \bar{q}(x) = \sum_{i=0}^{n} \bar{b}_i x^i,$$

where

$$\bar{a}_i = \begin{cases} a_i & \text{for } i \text{ even;} \\ -a_i & \text{for } i \text{ odd;} \end{cases} \quad \text{and} \quad \bar{b}_i = \begin{cases} b_i & \text{for } i \text{ even;} \\ -b_i & \text{for } i \text{ odd.} \end{cases}$$

Show that $[(\bar{q}(x))/(\bar{p}(x))]$ approximates e^x equally well; *i.e.*,

$$\bar{q}(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^N}{N!} + O(x^{N+1}).$$

-
p (x)

For example, we find

$$\frac{720+600x+240x^2+60x^3+10x^4+x^5}{720-120x} = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\frac{x^6}{6!}+O(x^7);$$

$$\frac{720+120x}{720-600x+240x^2-60x^3+10x^4-x^5} = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\frac{x^6}{6!}+O(x^7).$$

Solution by the proposer

We remark that the rational approximation of e^x given in the problem is an instance of a *Padé* approximation; and in the special case of the exponential function the Padé approximation of bidegree (m,n) is explicitly known [1, page 24]:

$$e^x \approx \frac{p(x)}{q(x)} = \frac{\sum_{i=0}^{m} \binom{m}{i} (m+n-i)!\, x^i}{\sum_{i=0}^{n} \binom{n}{i} (m+n-i)!\, (-x)^i},$$

from which the assertion of the problem follows at once. We can prove a somewhat more general result:

Proposition. Suppose ϕ is an analytic function in some neighborhood of 0 and for x in that neighborhood

$$\phi(x)=1+x+\frac{x^2}{}+\frac{x^3}{}+\frac{x^4}{}+\ldots+\frac{x^n}{}+\left[\frac{1}{}-\varepsilon\right]x^{n+1}+\ldots.$$

2! 3! 4! n! ⌊ (n+1)! ⌋

Then with $\psi(x)=[\,1/(\phi(-x))]$ we have, in a neighborhood of 0,

$$\psi(x)=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots+\frac{x^n}{n!}+\left[\frac{1}{(n+1)!}-(-1)^n\varepsilon\right]x^{n+1}+\ldots.$$

Before proving this proposition, we remark that there are indeed such approximations of e^x which are not rational functions, for example (the silly)

$$e^x\approx\sqrt{\frac{1+x}{1-x}}=1+x+\frac{x^2}{2}+\frac{x^3}{2}+\ldots=1+x+\frac{x^2}{2!}+\left[\frac{1}{3!}-\left(-\frac{1}{3}\right)\right]x^3+\ldots.$$

Also the proposition gives the assertion of the problem, where

$$\phi(x)=\frac{p(x)}{q(x)}\text{, and }\psi(x)=\frac{\bar{q}(x)}{\bar{p}(x)}=\frac{q(-x)}{p(-x)}=\frac{1}{\phi(-x)}.$$

Proof of proposition. First we observe the well-known

$$1\le i\Rightarrow\sum_{j=0}^{i}(-1)^j\binom{i}{j}=\sum_{j=0}^{i}\binom{i}{j}1^{i-j}(-1)^j=[1+(-1)]^i=0^i=0.\qquad(*)$$

Then we recall Leibniz's formula, easily proved by induction:
Suppose for $0\le i$ that there exist i derivatives of u and of v ; then there exist i derivatives of the product uv , and moreover $(uv)^{(i)}=\sum_{j=0}^{i}(i\,\|\,j)\,u^{(i-j)}v^{(j)}$ (**)
where $^{(k)}$ denotes k-th derivative.
By hypothesis $\phi(-x)\psi(x)=1$, at least for x near 0. Clearly $0\le i\le n\Rightarrow\phi^{(i)}(0)=1$; we shall show that the same holds for $\psi^{(i)}(0)$.

Observe first that from $\phi(0)\psi(0)=1$ we have $\psi(0)=1$; assume (inductively) $1 \le i \le n$, and $0 \le j < i \Rightarrow \psi^{(j)}(0)=1$. By the chain rule $0 \le i \Rightarrow [\phi(-x)]^{(i)}=(-1)^i\phi^{(i)}(-x)$.
By (**)

$$1 \le i \le n \Rightarrow [\phi(-x)\psi(x)]^{(i)}=1^{(i)}=0 \;= \sum_{j=0}^{i} \binom{i}{j} [\phi(-x)]^{(i-j)}\psi^{(j)}(x)$$

$$= \sum_{j=0}^{i} \binom{i}{j} (-1)^{i-j}\phi^{(i-j)}(-x)\psi^{(j)}(x).$$

Then with x=0 we find, since $\phi^{(i-j)}(0)=1$, that $\sum_{j=0}^{i}(i \,\|\, j)(-1)^{i-j}\psi^{(j)}(0)=0$.

Hence

$$1 \le i \le n \Rightarrow \psi^{(i)}(0) \;=(-1)^{i-1} \sum_{j=0}^{i-1} (-1)^j \binom{i}{j} \psi^{(j)}(0)$$

$$=(-1)^{i-1} \sum_{j=0}^{i-1} (-1)^j \binom{i}{j} \quad \text{by inductive hypothesis}$$

$$=(-1)^{i-1}\cdot(-1)^{i-1} \quad \text{by (*)}$$

$$=1.$$

Thus

$$0 \le i \le n \Rightarrow \psi^{(i)}(0)=1. \qquad (***)$$

By (**) again we have

$$\sum^{n+1} \binom{n+1}{j} [\phi(-x)]^{(n+1-j)}\psi^{(j)}(x)=0,$$

$j=0$

whence

$$\begin{aligned}\psi^{(n+1)}(0) &= -(-1)^{n+1}\phi^{(n+1)}(0) - \sum_{j=1}^{n} \binom{n+1}{j} (-1)^{n+1-j} \\ &= (-1)^{n}\phi^{(n+1)}(0) - (-1)^{n+1} \sum_{j=1}^{n} (-1)^{j} \binom{n+1}{j} \\ &= (-1)^{n}\phi^{(n+1)}(0) - (-1)^{n+1}\cdot[-1-(-1)^{n+1}] \\ &= (-1)^{n}\phi^{(n+1)}(0) + (-1)^{n+1} + 1.\end{aligned}$$

from which one easily completes the proof of the proposition.

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[1] D.J.Newman, *Approximation with Rational Functions*, Conference Board of the Mathematical Sciences, Number 41, American Mathematical Society, Providence, Rhode Island, 1978