Problem for 1996 November

Proposed by Dan Jurca

Suppose that n is a non-negative integer, $r \in \mathbf{R}$, and $1 < |r|$; then (by the ratio test) $\sum_{i=1}^{\infty} [(i^n)/(r^i)]$ converges; find a formula, or practical method, to evaluate the sum precisely.

Solution by the proposer

For a fixed $r \in \mathbf{R}$, $1 < |r|$ we write, for non-negative integers n: $S_n=\sum_{i=1}^{\infty}[(i^n)/(r^i)]$. Then

$$S_0=\sum_{i=1}^{\infty} \frac{1}{r^i} = \frac{1/r}{1-1/r} = \frac{1}{r-1} .$$

Now suppose $1 \le n$; we shall compute S_n in terms of $S_0, S_1,\dots,S_{n-1}$ as follows.

$$S_n=\sum_{i=1}^{\infty} \frac{i^n}{r^i}$$

so that

$$\begin{aligned} rS_n &= \sum_{i=1}^{\infty} \frac{i^n}{r^{i-1}} \\ &= \sum_{i=0}^{\infty} \frac{(i+1)^n}{r^i} \\ &= \sum_{i=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} i^j \cdot \frac{1}{r^i} \end{aligned}$$

$$= \sum_{j=0}^{n} \binom{n}{j} \sum_{i=0}^{\infty} \frac{i^j}{r^i}$$

$$= \sum_{j=0}^{n} \binom{n}{j} S_j$$

$$= \sum_{j=0}^{n-1} \binom{n}{j} S_j + S_n.$$

Therefore

$$(r-1)S_n = \sum_{i=0}^{n-1} \binom{n}{i} S_i, \text{ whence}$$

$$S_n = \frac{1}{r-1} \cdot \sum_{i=0}^{n-1} \binom{n}{i} S_i,$$

from which each S_n may be computed recursively.