Problem for 1997 May

Proposed by Dan Jurca

Suppose f:**R**→**R** is defined by f(t)=t and

$$I=\int_0^{\int_0^{\int_0^{\cdots} f} f} f$$

exists and is a positive real number.

What is the value of I?

(You will have to find a ``reasonable'' *definition* of the integral above.)

Solution by the proposer

Let x be a real number, and consider the sequence $(I_n)_{n=0}^{\infty}$ defined as follows:

$$I_n=\begin{cases} x & \text{if } n=0; \\ \int_0^{I_{n-1}} f & \text{if } 1 \le n. \end{cases}$$

Thus

$$I_0 = x$$

$$I_1 = \int_0^x f$$

$$I_2 = \int^{\int_0^x f} f$$

$$I_3 = \int_0^{\int_0^{\int_0^x f} f} f$$

$$\vdots$$

Then we define the integral in the statement of the problem as the limit of (I_n) if this limit exists; otherwise, we do not define the integral. An easy calculation shows that

$$0 \leq n \Rightarrow I_n = 2(x/2)^{2n}.$$

(This is also easily proved by induction from the definition of I_n.)

Now clearly (I_n) diverges if $2 < |x|$, and (I_n) converges to 0 if $|x| < 2$. However, if $x=\pm 2$, then (I_n) converges to 2. Therefore we say $I=2$.