Problem for 1997 September and October

Let **Q** be the set of rational numbers, so $\mathbf{Q} \subset \mathbf{R}$, where **R** is the set of real numbers. Let $\mathbf{R}^2$ be topologized as usual, and consider $S=(\mathbf{Q}\times(\mathbf{R}-\mathbf{Q}))\cup((\mathbf{R}-\mathbf{Q})\times\mathbf{Q}) \subset \mathbf{R}^2$.
Thus S is the set of points in the Cartesian plane with one coordinate rational and the other coordinate irrational.

Prove that S is not connected.

Solution by Professor Emeritus Victor Manjarrez

The line with equation y=x is disjoint from S, and is a closed subset of $\mathbf{R}^2$; hence let $U=\{(x,y) \in \mathbf{R}^2 \mid x < y\}$ and $V=\{(x,y) \in \mathbf{R}^2 \mid y < x\}$. Then U is open in $\mathbf{R}^2$, V is open in $\mathbf{R}^2$, neither U nor V is empty, $U\cap V=\varnothing$, and $S \subset U\cup V$.

Therefore S is not connected.