Problem for 1998 July and August

proposed by Dan Jurca

For each continuous function $f:\mathbf{R}\to\mathbf{R}$ let $Tf:\mathbf{R}\to\mathbf{R}$ by

$$x \in \mathbf{R} \Rightarrow Tf(x)=\int_0^x f=\int_0^x f(u)du,$$

and for n = 2, 3, … let $T_n f$ be the function obtained by applying T n times to f. For example, if $g(t)=t^2$, then

$$Tg(x) = \int_0^x g=\int_0^x g(u)\,du=\int_0^x u^2\,du=x^3/3, \text{ and}$$

$$T_2g(x) = TTg(x)=\int_0^x Tg=\int_0^x Tg(u)\,du=\int_0^x u^3/3\,du=x^4/12.$$

Now let $f(x)=e^x$, and determine $\lim_{n\to\infty}T_n f$.

Here is another (equivalent) way to state the problem.

Let $C(\mathbf{R})$ be the (infinite-dimensional) real vector space of all continuous functions $\mathbf{R}\to\mathbf{R}$ and let $T:C(\mathbf{R})\to C(\mathbf{R})$ by

$$f \in C(\mathbf{R}) \Rightarrow Tf \in C(\mathbf{R}) \text{ by } Tf(x)=\int_0^x f=\int_0^x f(u)\,du.$$

Then for n=0, 1, 2, … there is a linear operator $T_n:C(\mathbf{R})\to C(\mathbf{R})$ defined as follows:

$$T_0 = \mathrm{id}_{C(\mathbf{R})};$$
$$1 \le n \Rightarrow T_n = T \circ T_{n-1}.$$

If $f \in C(\mathbf{R})$ by $f(x)=e^x$, what is $\lim_{n\to\infty} T_n f$?

(Observe that we are asking for $\lim_{n\to\infty}(T_n f)$.)

Solution by the proposer

We find at once

$$T_0 f(x) = f(x)$$
$$T_1 f(x) = \int_0^x T_0 f = \int_0^x e^u du = e^x - 1$$
$$T_2 f(x) = \int_0^x T_1 f = \int_0^x [e^u - 1] du = e^x - x - 1 = e^x - (1+x)$$
$$T_3 f(x) = \int_0^x T_2 f = \int_0^x [e^u - (1+u)] du = \ldots = e^x - (1 + x + \frac{x^2}{2}).$$

Now let us write $p_{-1}(x)=0$, and, for $n=0,1,2,\ldots$,

$$p_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}.$$

Then by induction on n we find easily

$$0 \le n \Rightarrow T_n f(x) = e^x - p_{n-1}(x).$$

Now as everyone knows $x \in \mathbf{R} \Rightarrow$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{x^k}{k!} = \lim_{n\to\infty} p_n(x).$$

It follows at once that $x \in \mathbf{R} \Rightarrow$

$$\lim_{n\to\infty} T_n f(x) = \lim_{n\to\infty} (e^x - p_{n-1}(x)) = e^x - \lim_{n\to\infty} p_{n-1}(x) = e^x - \lim_{n\to\infty} p_n(x) = e^x - e^x = 0.$$

Thus we have: $x \in \mathbf{R} \Rightarrow \lim_{n\to\infty} T_n f(x) = 0$, and since this holds for each x, we write $\lim_{n\to\infty} T_n f = 0$.

Remark: Although the argument above holds for the particular function $f(x)=e^x$, in fact one can show that the same result holds for each continuous $f:\mathbf{R}\to\mathbf{R}$.