Problem for 1999 February

Proposed by Professor Emeritus Victor Manjarrez

Prove that if A is a finite subset of **C**, the set of complex numbers, and if

$$\left|\sum_{a \in A} a\right| = \sum_{a \in A} |a|,$$

then for each subset B of A

$$\left|\sum_{b \in B} b\right| = \sum_{b \in B} |b|.$$

Solution by the proposer

If C=A\B, then

$$\begin{aligned}
\sum_{a \in A} |a| &= \left|\sum_{a \in A} a\right| \\
&= \left|\sum_{b \in B} b + \sum_{c \in C} c\right| \\
&\le \left|\sum_{b \in B} b\right| + \left|\sum_{c \in C} c\right| \\
&\le \sum_{b \in B} |b| + \left|\sum_{c \in C} c\right| \\
&\le \sum_{b \in B} |b| + \sum_{c \in C} |c|
\end{aligned}$$

$$= \sum_{a \in A} |a|.$$

It follows that all these are equal, so that

$$\left|\sum_{b \in B} b\right| + \left|\sum_{c \in C} c\right| = \sum_{b \in B} |b| + \left|\sum_{c \in C} c\right| ,$$

whence $|\sum_{b \in B} b| = \sum_{b \in B} |b|$, as asserted.

Remark: The assertion and proof are immediately generalized to normed spaces.