Problem for 1999 May

Communicated by Dan Jurca

Determine whether

$$\lim_{n\to\infty} \frac{n}{2^n} \sum_{k=1}^{n} \frac{2^k}{k}$$

exists; and, if so, determine the limit.

Solution by Dan Jurca

The limit exists and equals 2. Write $x_n=[\, n/(2^n)]\sum_{k=1}^{n}[(2^k)/k]$. Then

$$2 \le n \Rightarrow x_n = \frac{n}{2^n}\left(\sum_{k=1}^{n-1} \frac{2^k}{k} + \frac{2^n}{n} \right) = \frac{n}{2^n}\,\frac{2^{n-1}}{n-1}\, x_{n-1} + 1$$

so that

$$x_n = \begin{cases} 1 & \text{if } n=1; \\ 1 + \dfrac{n}{2(n-1)}\, x_{n-1} & \text{if } 2 \le n. \end{cases}$$

We next show that $(x_n)_{n=5}^{\infty}$ is a decreasing sequence; *i.e.*, $6 \le n \Rightarrow x_n < x_{n-1}$. Clearly $x_6=13/5 < 8/3=x_5$, so this holds if n=6. Now suppose that $7 \le n$ and $x_{n-1} < x_{n-2}$. Then we have

$$n^2-2n < n^2-2n+1, \text{ so}$$

$$n(n-2) < (n-1)^2, \text{ so}$$

$$\frac{n}{} < \frac{n-1}{}, \text{ and}$$

$$n-1 \qquad n-2$$

$$\frac{n}{2(n-1)} < \frac{n-1}{2(n-2)}, \text{ so, since } \forall k: 0 < x_k,$$

$$\frac{n}{2(n-1)} x_{n-1} < \frac{n-1}{2(n-2)} x_{n-2}, \text{ whence}$$

$$x_n < x_{n-1}.$$

It follows by induction that $6 \leq n \Rightarrow x_n < x_{n-1}$, so $(x_n)_{n=5}^{\infty}$ is a decreasing sequence, as asserted. Since $(x_n)_{n=5}^{\infty}$ is obviously bounded below, it follows that $(x_n) \to L$, some $L \in \mathbf{R}$. But then from

$$2 \leq n \Rightarrow x_n = 1 + \frac{n}{2(n-1)} x_{n-1}$$

it follows that

$$L = 1 + \frac{1}{2} L$$

so that L=2, as asserted.