Problem for 1999 June

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Let **N**={1,2,3,…}, the set of natural numbers, and *P*={2,3,5,7,11,…}, the set of prime numbers. For each subset $S \subset P$ let

$$N(S)=\{n \in \mathbf{N} \mid p \in P \text{ and } p|n \Rightarrow p \in S\}.$$

Equivalently,

$$N(S)=\{\prod_{i=1}^{n} p_i^{k_i} \mid 0 \le n,\ p_i \in S, \text{ and } 0 \le k_i\}.$$

Thus N(S) is the set of positive integers the only prime divisors of which are in S. Thus, for example, $S \subset P \Rightarrow 1 \in N(S)$, even if $S=\emptyset$. Again,

$$N(\{2,3\})=\{1,2,3,4,6,8,9,12,16,18,24,27,32,36,\dots\}. \qquad (*)$$

1. Compute

$$R(\{2,3\})=\sum_{n \in N(\{2,3\})} \frac{1}{n}.$$

I.e., determine the sum of the reciprocals of the integers in the set (*).

2. Show that if S is finite, then

$$R(S)=\sum_{n \in N(S)} \frac{1}{n} < \infty;$$

that is, if S is a finite subset of *P*, then the sum of the reciprocals of all the positive integers whose only prime divisors are in S exists.
(Since $\sum_{n=1}^{\infty} 1/n$ diverges, this shows that *P* is infinite.)

3. Find a formula for R(S), where $S \subset P$ and S is finite.

Remark-It can be shown that there exists an infinite S (and hence uncountably many infinite S) such that $R(S) < \infty$.