

Problem for 1999 August

Proposed by Dan Jurca

The Fibonacci sequence

$$(F_n)_{n=0}^{\infty} = (0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \dots)$$

may be defined as follows.

$$F_0 = 0;$$

$$F_1 = 1;$$

$$2 \leq n \Rightarrow F_n = F_{n-2} + F_{n-1}.$$

1.

Find the sum of the first n terms of $(F_i)_{i=0}^{\infty}$; *i.e.*, evaluate

$$\sum_{i=0}^{n-1} F_i = F_0 + F_1 + \dots + F_{n-1}.$$

2.

Find the sum of the first n terms of $(F_i^2)_{i=0}^{\infty}$; *i.e.*, evaluate

$$\sum_{i=0}^{n-1} F_i^2 = F_0^2 + F_1^2 + \dots + F_{n-1}^2.$$

Solution by the proposer

It is very easy to discover, and then to prove by induction, that

$$1 \leq n \Rightarrow \sum_{i=0}^{n-1} F_i = F_{n+1} - 1 \quad \text{and}$$

$$\sum_{i=0}^{n-1} F_i^2 = F_{n-1} F_n.$$

