Problem for 2000 April

proposed by Dan Jurca

Find the sum of the first n terms of the following series.

$$1^2+2^2+4^2+7^2+11^2+16^2+22^2+29^2+37^2+46^2+56^2+\ldots$$

Solution by the proposer

We write S_n for the sum of the first n terms; then

$$S_n = \sum_{i=1}^{n} a_i^2, \quad \text{where}$$

$$a_i = \frac{i^2-i+2}{2}.$$

(The differences between consecutive a_i make an arithmetic series, so that a_i equals some quadratic in i; one finds easily that the above formula holds for a_i.)
Hence

$$S_n = \sum_{i=1}^{n} \left[\frac{i^2-i+2}{2} \right]^2$$

$$= \frac{1}{4} \sum_{i=1}^{n} (i^4-2i^3+5i^2-4i+4)$$

$$= \frac{1}{4} \sum_{i=1}^{n} i^4 - \frac{1}{2} \sum_{i=1}^{n} i^3 + \frac{5}{4} \sum_{i=1}^{n} i^2 - \sum_{i=1}^{n} i + \sum_{i=1}^{n} 1$$

$$= \frac{6n^5+15n^4+10n^3-n}{120} - \frac{n^4+2n^3+n^2}{8} + \frac{10n^3+15n^2+5n}{24} - \frac{n^2+n}{2} + n$$

$$= \frac{6n^5+30n^3+84n}{120}$$

$$= \frac{n^5+5n^3+14n}{20}.$$

Also solved by William Loewe, John Sayer, and Professor Ed Keller