Problem for 2000 July

proposed by Dan Jurca

The n-simplex $\triangle^n$ may be defined as follows.

$$\triangle^n=\{(x_0,x_1,\ldots,x_n) \in \mathbf{R}^{n+1} \mid 0 \le i \le n \Rightarrow 0 \le x_i, \text{ and } x_0+x_1+\ldots+x_n=1\}$$

The sketch below shows $\triangle^0$=A, $\triangle^1$=AB, $\triangle^2$=ABC; $\triangle^3$ is a tetrahedron in $\mathbf{R}^4$.

Let us say that the 0-volume of a finite set of points is the number of points in the set, the 1-volume of a line segment is the length of the segment, the 2-volume of a polygonal region is the area of the region, the 3-volume of a polyhedron is the ordinary volume, *etc.*

Find a formula for the n-volume of $\triangle^n$.

Solution by the proposer

Writing V_n for the n-volume of $\triangle^n$ we show by induction on n that $0 \le n \Rightarrow V_n=\sqrt{n+1}/n!$. This clearly holds if n=0 (for $\triangle^0$ is a set consisting of a single point), or n=1 (for $\triangle^1$ is a line segment of length $\sqrt{2}$), or n=2 (for $\triangle^2$ is the closed region bounded by an equilateral triangle of side $\sqrt{2}$). So suppose that $2 \le n$ and $V_{n-1}=\sqrt{n}/(n-1)!$. The following figure

FIGURE NOT AVAILABLE

shows that we may consider $\triangle^n$ as an n-dimensional ``pyramid'' with base $\triangle^{n-1}$ and height, say, h_n. (The figure shows $\triangle^3$ as a tetrahedron with each side of length $\sqrt{2}$, base $\triangle^2$ an equilateral triangle with sides of length $\sqrt{2}$, and height h_3). Now $\triangle^n$ may be considered made up of slices which look like $\triangle^{n-1}$; the (n–1)-volume of such a slice at distance x from the apex of $\triangle^n$ is equal to $(x/h_n)^{n-1}\times V_{n-1}$. (This clearly holds if x=0 or $x=h_n$; for $0 < x < h_n$ it holds almost by definition of (n–1)-volume-the (n–1)-volume of an (n–1)-dimensional cube is the (n–1)-th power of the length of an edge of the cube.) Therefore we have

$$1 \le n \Rightarrow V_n = \int^{h_n} \left(x \right)^{n-1} V_{n-1}\thinspace dx$$

$$\int \; 0 \quad (\quad h_n \;)$$

$$= \frac{1}{n} h_n V_{n-1}.$$

To complete the argument by induction we need to determine h_n. Now h_n is the distance from the point $(0,0,\ldots,0,1) \in \mathbf{R}^{n+1}$ to \triangle^{n-1}, which may clearly be assumed to be a subset of $\mathbf{R}^{n+1}$ as well as of $\mathbf{R}^n$. Thus h_n is the distance from $(0,0,\ldots,0,1)$ to \triangle^{n-1}$=\{(x_0,x_1,\ldots,x_{n-1},0) \mid 0 \leq x_i$ and $x_0+x_1+\ldots+x_{n-1}=1\}$. We may compute this distance by minimizing the square of the distance from the apex, $(0,0,\ldots,0,1)$, to the points in \triangle^{n-1}. So consider

$$\begin{aligned} f(x_0,x_1,\ldots,x_{n-2}) &= x_0^2+x_1^2+\ldots+x_{n-1}^2+1 \\ &= x_0^2+x_1^2+\ldots+(1-x_0-x_1-\ldots-x_{n-2})^2+1. \end{aligned}$$

Computing the partial derivatives of f with respect to the x_i we have

$$\begin{aligned} \frac{\partial f}{\partial x_i}(x_0,x_1,\ldots,x_{n-2}) &= 2x_i+2(1-x_0-x_1-\ldots-x_{n-2})\cdot -1 \\ &= 2x_i-2+2x_0+2x_1+\ldots+2x_{n-2}, \end{aligned}$$

so setting each of these n–1 partial derivatives to zero yields the following $(n-1)\times(n-1)$ linear system.

$$\begin{bmatrix} 2 & 1 & 1 & \ldots & 1 \\ 1 & 2 & 1 & \ldots & 1 \\ 1 & 1 & 2 & \ldots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \ldots & 2 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{n-2} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

The coefficient matrix, say A, above is nonsingular. In fact the determinant equals n. This may be verified by observing that A has the following n–1 linearly independent eigenvectors.

$$
\begin{aligned}
\mathbf{v}_1 &=(1,1,1,\ldots,1)^T\\
\mathbf{v}_2 &=(1,-1,0,0,\ldots,0,0)^T\\
\mathbf{v}_3 &=(1,0,-1,0,\ldots,0,0)^T\\
\mathbf{v}_4 &=(1,0,0,-1,\ldots,0,0)^T\\
&\vdots\\
\mathbf{v}_{n-1} &=(1,0,0,0,\ldots,0,-1)^T,
\end{aligned}
$$

with associated eigenvalues n,1,1,1,…,1. Therefore there exists a unique solution of the linear system; by inspection it is $x_i=1/n$, for i=0,1,…,n–2. It follows that h_n, the distance from the apex (0,0,…,0,1) of $\triangle^n$ to its base $\triangle^{n-1}$, is equal to the distance in $\mathbf{R}^{n+1}$ from the point (0,0,…,0,1) to the point (1/n,1/n,…,1/n,0), which equals

$$
\begin{aligned}
h_n &= \sqrt{(1/n)^2+(1/n)^2+\ldots+(1/n)^2+1}\\
&= \sqrt{1/n+1}\\
&= \sqrt{\frac{n+1}{n}}\ .
\end{aligned}
$$

We can now complete the inductive argument.

$$
\begin{aligned}
V_n &= \frac{1}{n} h_n V_{n-1}\\
&= \frac{1}{n} \times \sqrt{\frac{n+1}{n}} \times \frac{\sqrt{n}}{(n-1)!}\\
&= \frac{\sqrt{n+1}}{n!},
\end{aligned}
$$

as asserted.

It may be interesting to observe that $(V_n)_{n=0}^{\infty} \to 0$ quite rapidly, and $(h_n)_{n=1}^{\infty} \to 1$.

Also solved by John M. Sayer