Problem for 2000 November

Proposed by Dan Jurca

For n=0, 1, 2, 3, … the Pascal matrix P_n is the (n+1)×(n+1) matrix

$$P_n=(p_{ij})_{0 \le i \le n, 0 \le j \le n} \quad \text{where } p_{ij} \text{ is the binomial coefficient } \binom{i+j}{i}.$$

For example,

$$P_4=\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 70 \end{pmatrix}$$

Compute the determinant $\det(P_n)$ of P_n.

Solution by Professor Bill Nico

Row reduce from bottom up; *i.e.*, subtract row n–1 from row n, then row n–2 from row n–1, *etc.*, and finally row 0 from row 1. Then column reduce from right to left; *i.e.*, subtract column n–1 from column n, then column n–2 from column n–1, *etc.*, and finally column 0 from column 1. To see the result, let Q_n be the following (n+1)×(n+1) matrix.

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 1 & \dots & 0 & 0 & 0 \end{pmatrix}$$

$$
\begin{pmatrix}
\cdots & \cdots & \ddots & \ddots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \ddots & 1 & 0 & 0 \\
0 & 0 & 0 & \cdots & -1 & 1 & 0 \\
0 & 0 & 0 & \cdots & 0 & -1 & 1
\end{pmatrix}
$$

Then we claim

$$Q_nP_nQ_n^T = \begin{pmatrix} 1 & \mathbf{0}_{1\times n} \\ \mathbf{0}_{n\times 1} & P_{n-1} \end{pmatrix}.$$

This follows since if c_{ij} is the element in row i and column j of $Q_nP_nQ_n^T$, then

$$
\begin{aligned}
c_{ij} &= p_{i-1,j-1} - p_{i-1,j} - p_{i,j-1} + p_{ij} \quad \text{and} \\
-p_{i-1,j} - p_{i,j-1} + p_{ij} &= -\binom{i+j-1}{i-1} - \binom{i+j-1}{i} + \binom{i+j}{i} \\
&= 0.
\end{aligned}
$$

Since $\det Q_n = \det Q_n^T = 1$, it follows at once that $\det(P_n) = \det(P_{n-1})$. Since $\det(P_0) = 1$, it therefore follows that $0 \le n \Rightarrow \det(P_n) = 1$.

Also solved by Walter Becker, Matthew Hubbard, and the proposer.