Problem for 1999 December 2000 January

Proposed by Dan Jurca

Part a. here is problem 1978 from the 1994 October issue of *Crux Mathematicorum*.

a. An experimenter E tosses a fair coin until the coin comes up heads for the first time, and counts that exactly k tosses were made; E then chooses randomly an integer from 1 to k. What is the probability that E chooses the number 1?

b. E begins anew, but now tosses until the coin comes up heads for the second time, and counts that exactly k tosses were made; again E chooses randomly an integer from 1 to k. What is the probability that E chooses the number 1?

c. Generalize. Suppose that n is a positive integer; suppose that E tosses a fair coin until it comes up heads for the n-th time, and E observes that exactly k tosses were made; again E chooses randomly an integer from 1 to k. As a function of n what is the probability P_n that E chooses the number 1?

Solution by the proposer

The following is a solution of part c. Parts a. and b. are special cases.

Suppose that $1 \le n$. If the coin comes up heads for the n-th time on the k-th toss, then in the first $k-1$ tosses there were exactly $n-1$ heads and hence $k-n$ tails. The number of sequences consisting of $n-1$ heads and $k-n$ tails is given by the following expression.

$$\binom{k-1}{n-1}$$

Thus the probability that the coin comes up heads for the n-th time on the k-th toss is

$$\binom{k-1}{n-1}\left(\frac{1}{2}\right)^k$$

so that the probability that the coin comes up heads for the n-th time on the k-th toss *and* that E chooses the number 1 is

$$\binom{k-1}{n-1}\left(\frac{1}{2}\right)^k \times \frac{1}{k};$$

so that the probability that E chooses the number 1 is

$$P_n = \sum_{k=n}^{\infty} \binom{k-1}{n-1} \frac{1}{k} \left(\frac{1}{2} \right)^k .$$

We next compute this sum. First we observe that

$$P_n = \sum_{i=0}^{\infty} \binom{n-1+i}{n-1} \frac{1}{n+i} \left(\frac{1}{2} \right)^{n+i} .$$

Now consider

$$f_n(x) = \sum_{i=0}^{\infty} \binom{n-1+i}{n-1} x^{n-1+i}$$

and observe that

$$\int_0^{1/2} f_n = \sum_{i=0}^{\infty} \binom{n-1+i}{n-1} \frac{1}{n+i} x^{n+i} \Big|_0^{1/2}$$

so that

$$P_n = \int_0^{1/2} f_n .$$

Now it is well-known [1, page 199] that

$$\sum_{i=0}^{\infty} \binom{m+i}{m} z^i = \frac{1}{(1-z)^{m+1}} ,$$

so that

$$f_n(x)=\frac{x^{n-1}}{(1-x)^n}$$

Therefore

$$P_n=\int_0^{1/2}\frac{x^{n-1}}{(1-x)^n}\,dx, \qquad 1\le n.$$

This integral is easily computed. We have

$$P_1=\int_0^{1/2}\frac{1}{1-x}\,dx=\ln 2,$$

and if $2\le n$, then

$$u=x^{n-1},\ dv=\frac{dx}{(1-x)^n} \Rightarrow du=(n-1)x^{n-2}dx,\ v=\frac{1}{n-1}\ \frac{1}{(1-x)^{n-1}}$$

so that

$$2\le n\Rightarrow P_n \ =\frac{1}{n-1}\ \frac{x^{n-1}}{(1-x)^{n-1}}\ \Big|_0^{1/2} - \int_0^{1/2}\frac{x^{n-2}}{(1-x)^{n-1}}\,dx$$

$$=\frac{1}{n-1}-P_{n-1}.$$

To eliminate the recursion here we write $A_0=0$, and

$$1\le n\Rightarrow A_n=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+-\ldots+\frac{(-1)^{n-1}}{n},$$

the n-th partial sum of the alternating harmonic series. Then one easily proves by induction that

$$1 \le n \Rightarrow P_n=(-1)^n(A_{n-1}-\ln 2).$$

Now writing $H_n=1+1/2+1/3+1/4+\ldots+1/n$, the n-th partial sum of the harmonic series, we have at once $A_n=H_n-H_{\lfloor n/2 \rfloor}$. Since it is known [1, page 264] that

$$1 \le n \Rightarrow H_n=\ln n+\gamma+\frac{1}{2n}-\frac{1}{12n^2}+\frac{\varepsilon_n}{120n^4}, \quad 0 < \varepsilon_n < 1, \quad \gamma = 0.5772156649\ldots,$$

we can, after some tedious but straightforward calculation, deduce, for example, that

$$P_n \sim \frac{1}{2n}; \text{ i.e., } \lim_{n\to\infty} \frac{P_n}{1/(2n)}=1,$$

which probably agrees well with what one would expect. That is, one might expect that heads will turn up for the n-th time after about 2n tosses, so that for large n we expect k to be about 2n; hence the probability of choosing the number 1 is approximately 1/(2n).

[1]: *Concrete Mathematics*, by Graham, Knuth, and Patashnik

One other solution was received, in which the solver considered the case of an unfair coin as well. Unfortunately, this solution was misplaced. It is hoped that the solver will resubmit the solution.