Problem for 2001 February

Proposed by Dan Jurca

The largest circle in the sketch below has a radius of 1; inside it and tangent to it is a circle of radius 1/2; then there is a circle tangent to these two with radius 1/3; finally there is a circle with radius 1/4 tangent to the largest one and the one with radius 1/3. Suppose this construction is repeated indefinitely; is there, in any sense, a limiting position? Or do the circles keep going on forever?

NO FIGURE AVAILABLE

Solution by the proposer

For $1 \le n$ let C_n be the center of the circle with radius 1/n, and for $3 \le n$ let θ_n be the angle defined by $C_{n-1}C_1C_n$. (For example, θ_3 is the angle at the center of the biggest circle subtended by radii to the centers of the circles with radii 1/2 and 1/3.) Then for $3 \le n$ there is a triangle with sides $1-1/(n-1)$, $1-1/n$, and $1/(n-1)+1/n$, where θ_n is the angle opposite the side of length $1/(n-1)+1/n$. By the law of cosines we have

$$\left(\frac{1}{n-1}+\frac{1}{n}\right)^2 = \left(1-\frac{1}{n-1}\right)^2 + \left(1-\frac{1}{n}\right)^2 - 2\left(1-\frac{1}{n-1}\right)\left(1-\frac{1}{n}\right)\cos\theta_n$$

and solving for $\cos\theta_n$ we find

$$\cos\theta_n = 1-\frac{2}{(n-1)(n-2)}\text{ ,from which}$$

$$\sin\theta_n = \frac{2}{(n-1)(n-2)}\sqrt{n^2-3n+1}\ .$$

Now from

$3 \le n$ we get

$$n^2 - 4n + 4 \leq n^2 - 3n + 1 \text{ so that}$$

$$n - 2 \leq \sqrt{n^2 - 3n + 1} \text{ ;therefore}$$

$$\frac{2}{n-1} \leq \frac{2}{(n-1)(n-2)} \sqrt{n^2 - 3n + 1} \text{ , whence}$$

$$\frac{2}{n-1} \leq \sin\theta_n.$$

But since $0 \leq \theta \Rightarrow \sin\theta \leq \theta$, we have

$$3 \leq n \Rightarrow \frac{2}{n-1} \leq \theta_n.$$

It follows (from the divergence of the harmonic series) that $\sum_{n=3}^{\infty}\theta_n$ diverges; hence the sequence $(C_n)_{n=1}^{\infty}$ of the centers of the circles does not converge to a point in the plane.

A simpler argument is based on the observation that, for $2 \leq n$, the length of the polygonal path from C_1 to C_n is

$$\frac{1}{2} + \left(\frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \ldots + \left(\frac{1}{n-1} + \frac{1}{n}\right) = 2H_n - \left(2 + \frac{1}{n}\right)$$

where $H_n = \sum_{i=1}^{n} {}^{1}/_{i}$. Again, since $H_n \rightarrow \infty$, $(C_n)_{n=1}^{\infty}$ diverges.

Also solved by Matthew Hubbard

Show also that $0 < a_n - e < [\ 1/((n+1)(n+1)!)]$.