Problem for 2001 May

Communicated by Dan Jurca

Prove

$$\lim_{n\to\infty} \left[\frac{1}{\sqrt{n}} \left(\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}} \right) \right] = \int_0^1 \frac{dx}{\sqrt{x}} = 2;$$

and use a similar method to determine the value of

$$\lim_{n\to\infty} \left[\frac{1}{n} \left((n+1)\times(n+2)\times\ldots\times(2n) \right)^{1/n} \right].$$

Solution by Dan Jurca

First we observe that

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0^+} \int_a^1 \frac{dx}{\sqrt{x}}$$

$$= \lim_{a\to 0^+} 2\sqrt{x}\Big|_a^1$$

$$= \lim_{a\to 0^+} (2-2\sqrt{a})$$

$$=2,$$

so that the integral exists. Next we observe that for each positive integer n the integral is approximated by

$$\frac{1}{n} \quad n \quad \frac{1}{\quad i/n},$$

$$\sum_{i=1} \sqrt{}$$

and in fact

$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \frac{1}{\sqrt{i/n}}$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \sqrt{\left(\frac{n}{i}\right)}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \sqrt{\left(\frac{1}{i}\right)},$$

so that

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{1}{\sqrt{i}} = \int_0^1 \frac{dx}{\sqrt{x}} = 2.$$

Next, let

$$p_n = \frac{1}{n} \left((n+1)\times(n+2)\times\ldots\times(2n) \right)^{1/n}$$

$$= \frac{1}{n} \left(n(1+\frac{1}{n})\times n(1+\frac{2}{n})\times\ldots\times n(1+\frac{n}{n}) \right)^{1/n}$$

$$= \frac{1}{n} \cdot n\left((1+\frac{1}{n})\times(1+\frac{2}{n})\times\ldots\times(1+\frac{n}{n}) \right)^{1/n}$$

so that $\ln p_n = \underline{1} \quad {}^{n} \quad \ln(1+ \underline{i}\,).$

$$n \sum_{i=1} \qquad n$$

Therefore

$$\lim_{n\to\infty} \ln p_n = \int_0^1 \ln(1+x)dx$$

$$= \Big[(x+1)\ln(x+1) - (x+1) \Big] \Big|_0^1$$

$$= 2\ln 2 - 2 - 0 + 1$$

$$= \ln 4 - 1$$

$$= \ln \frac{4}{e},$$

so that $\lim_{n\to\infty} p_n = \frac{4}{e}$.

Also solved by Yi Shen and Yang Li Tang.