Problem for 2001 July

Proposed by Dan Jurca

Communicated by Dan Jurca

For $0 \le n$ let

$$I_n = \int_0^1 x^n e^x dx.$$

Prove $I_n \sim e/n$; *i.e.*, show that

$$\lim_{n\to\infty} \frac{I_n}{e/n} = 1.$$

Solution by the proposer

Using integration by parts one finds

$$I_0 = e-1 = -1-(-1)e; \quad 1 \le n \Rightarrow I_n = e - nI_{n-1} = -nI_{n-1} + e.$$

We write $0 \le n \Rightarrow I_n = a_n - b_n e$; then $a_0 = -1 = b_0$, and

$$\begin{aligned} 1 \le n \Rightarrow a_n &= -na_{n-1}, \\ b_n &= -(1+nb_{n-1}). \end{aligned}$$

Solving these recurrences we have

$$\begin{aligned} 0 \le n \Rightarrow a_n &= (-1)^{n+1} n! \\ b_n &= (-1)^{n+1} n! \sum_{i=0}^{n} \frac{(-1)^i}{i!} \end{aligned}$$

$i=0$

as one checks by induction on n. Hence

$$0 \le n \Rightarrow I_n = (-1)^{n+1} n! \left[1 - e \sum_{i=0}^{n} \frac{(-1)^i}{i!} \right]$$

$$= (-1)^{n+1} n! e \left[e^{-1} - \sum_{i=0}^{n} \frac{(-1)^i}{i!} \right].$$

Now

$$e^{-1} - \sum_{i=0}^{n} \frac{(-1)^i}{i!} = \frac{(-1)^{n+1}}{(n+1)!} + \frac{(-1)^{n+2}}{(n+2)!} + \frac{(-1)^{n+3}}{(n+3)!} + \dots$$

$$= \frac{(-1)^{n+1}}{(n+1)!} \left[1 - \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} - + \dots \right]$$

$$= \frac{(-1)^{n+1}}{(n+1)!} s_n$$

where

$$1 - \frac{1}{n+2} = \frac{n+1}{n+2} < s_n < 1.$$

Hence

$$1 \le n \Rightarrow I_n = \frac{n!e}{(n+1)!} s_n$$

$$= \frac{e}{n+1} s_n$$

$$= \frac{e}{n} \cdot \frac{n}{n+1} s_n,$$

so that, finally,

$$1 \le n \Rightarrow \frac{n}{n+2} < \frac{I_n}{e/n} < \frac{n}{n+1},$$

from which the assertion follows by the squeeze theorem.
It is clear that the quantity $I_n/(e/n)$ may be bounded more accurately.

Also solved by Walt Becker and Yi Shen