

Problem for 2001 September

Proposed by Dan Jurca

An observer at distance h above the surface of a spherical planet with diameter d can see what fraction of the area of the planet?

Solution by the proposer

We recall that by a spherical zone we mean the portion of a sphere between two parallel planes; and that the area of a spherical zone of height w on a sphere of diameter d equals πdw . Next, from the figure below we see that

$$\begin{aligned}\frac{r}{r+h} &= \frac{r-w}{r} && \text{so that} \\ w &= \frac{hr}{h+r} \\ &= \frac{hd}{2h+d}.\end{aligned}$$

Hence the fraction of the area of a sphere of diameter d visible from a point h above the surface of the sphere equals $A(h)$, where

$$\begin{aligned}A(h) &= \frac{\pi d \cdot \frac{hd}{2h+d}}{\pi d^2} \\ &= \frac{h}{2h+d}.\end{aligned}$$

Then we observe that, as expected,

$$\begin{aligned}0 &\leq A(h) < \frac{1}{2}, \\ A(0) &= 0, \quad \text{and}\end{aligned}$$

$$\lim_{\infty} A = \frac{1}{2},$$

and we also observe that

$$A'(h) = \frac{d}{(2h+d)^2}, \quad \text{so that}$$

$$A'(0) = \frac{1}{d}, \quad \text{and, also as expected,}$$

$$\lim_{\infty} A' = 0.$$

This was also solved by Walter Becker and Richie Hom.