Problem for 2002 January

Proposed by Dan Jurca

Modified from a problem in the Canadian mathematics journal *Crux Mathematicorum*

Suppose there are two players, say A and B; A tosses a fair coins, and B tosses b fair coins; let P(a,b) be the probability that B gets more heads than A.

1. Find a formula for P(n,n).

2. Show that
   a. $0 \le P(n,n) < {}^1/_2$;
   b. $(P(n,n))_{n=0}^{\infty}$\nearrow; *i.e.*, the sequence (P(n,n)) increases;
   c. $\lim_{n\to\infty} P(n,n) = {}^1/_2$.

3. Determine P(n,n+1).

Solution by the proposer

The probability that in a toss of n fair coins, $1 \le n$, exactly i heads occur is

$$\binom{n}{i}\left(\frac{1}{2}\right)^n, \quad 0 \le i \le n.$$

Therefore the probability that in two tosses of n fair coins the same number of heads occurs both times is

$$\binom{n}{0}\left(\frac{1}{2}\right)^n \times \binom{n}{0}\left(\frac{1}{2}\right)^n + \binom{n}{1}\left(\frac{1}{2}\right)^n \times \binom{n}{1}\left(\frac{1}{2}\right)^n + \cdots + \binom{n}{n}\left(\frac{1}{2}\right)^n \times \binom{n}{n}\left(\frac{1}{2}\right)^n = \left(\frac{1}{2}\right)^{2n} \sum_{i=0}^{n} \binom{n}{i}^2$$

$$= \left(\frac{1}{2}\right)^{2n} \binom{2n}{n},$$

the last equality following from the 2001 March ``problem of the month''. It follows that the probability that in two tosses of n fair coins there are more heads in the second toss is

$$P(n,n) = \frac{1}{2}\left[1 - \left(\frac{1}{2}\right)^{2n} \binom{2n}{n}\right].$$

It is now clear that $0 \le n \Rightarrow 0 \le P(n,n) < 1/2$. Next, from

$$\frac{\binom{2n}{n}}{2^{2n}} = \frac{\frac{(2n)(2n-1)}{nn}\binom{2n-2}{n-1}}{2\cdot 2\cdot 2^{2n-2}} = \frac{2n-1}{2n}\,\frac{\binom{2n-2}{n-1}}{2^{2n-2}} < \frac{\binom{2n-2}{n-1}}{2^{2n-2}}$$

we see that the sequence $((^1/_2)^{2n}(2n \parallel n))_{n=0}^{\infty}$ decreases, so that the sequence $(P(n,n))$ increases. Finally from the recursion formula

$$\int_0^{\pi/2} \sin^0\theta\, d\theta = \frac{\pi}{2}; \qquad 2 \le n \Rightarrow \int_0^{\pi/2} \sin^n\theta\, d\theta = \frac{n-1}{n} \int_0^{\pi/2} \sin^{n-2}\theta\, d\theta,$$

which follows from an application of integration by parts, we have

$$\frac{2}{\ } \int^{\pi/2} \sin^{2n}\theta\, d\theta = \frac{(2n-1)\times(2n-3)\times\ldots\times 1}{\ } = \left(\frac{1}{\ }\right)^{2n} \binom{2n}{\ },$$

$$\pi \int_0 \qquad (2n)\times(2n-2)\times\ldots\times 2 \qquad \left(\; 2 \; \right) \qquad \left(\; n \; \right)$$

(Wallis's product), from which it follows that

$$\lim_{n\to\infty} \left(\frac{1}{2} \right)^{2n} \binom{2n}{n} = 0,$$

whence $\lim_{n\to\infty} P(n,n) = {}^1/_2$. In fact

$$P(n,n) \sim \frac{1}{2} \left[1 - \frac{1}{\sqrt{\pi n}} \left(1 - \frac{1}{8n} + \frac{1}{128n^2} - \ldots \right) \right].$$

Now for $0 \le a$ and $0 \le b$ we clearly have

$$P(a,b) = \sum_{i=0}^{a} \binom{a}{i} \left(\frac{1}{2} \right)^{a} \sum_{j=i+1}^{b} \binom{b}{j} \left(\frac{1}{2} \right)^{b}$$

$$= \left(\frac{1}{2} \right)^{a+b} \sum_{i=0}^{a} \binom{a}{i} \sum_{j=i+1}^{b} \binom{b}{j}, \quad \text{so that}$$

$$P(n,n+1) = \left(\frac{1}{2} \right)^{2n+1} \sum_{i=0}^{n} \binom{n}{i} \sum_{j=i+1}^{n+1} \binom{n+1}{j}$$

$$= \left(\frac{1}{2} \right)^{2n+1} \sum_{i=0}^{n} \binom{n}{i} \sum_{j=i+1}^{n+1} \left[\binom{n}{j-1} + \binom{n}{j} \right]$$

$$= \left(\frac{1}{2} \right)^{2n+1} \sum_{i=0}^{n} \binom{n}{i} \left[\sum_{j=i+1}^{n+1} \binom{n}{j-1} + \sum_{j=i+1}^{n+1} \binom{n}{j} \right]$$

$$=\left(\frac{1}{2}\right)^{2n+1}\sum_{i=0}^{n}\binom{n}{i}\left[\sum_{j=i}^{n}\binom{n}{j}+\sum_{j=i+1}^{n}\binom{n}{j}\right]$$

$$=\left(\frac{1}{2}\right)^{2n+1}\sum_{i=0}^{n}\binom{n}{i}\left[\binom{n}{i}+2\sum_{j=i+1}^{n}\binom{n}{j}\right]$$

$$=\left(\frac{1}{2}\right)^{2n+1}\left[\sum_{i=0}^{n}\binom{n}{i}^{2}+2\sum_{i=0}^{n}\binom{n}{i}\sum_{j=i+1}^{n}\binom{n}{j}\right]$$

$$=\left(\frac{1}{2}\right)^{2n+1}\left\{\binom{2n}{n}+\left[\sum_{i=0}^{n}\binom{n}{i}\right]^{2}-\sum_{i=0}^{n}\binom{n}{i}^{2}\right\}$$

$$=\left(\frac{1}{2}\right)^{2n+1}\left[\binom{2n}{n}+(2^{n})^{2}-\binom{2n}{n}\right]$$

$$=\left(\frac{1}{2}\right)^{2n+1}\times 2^{2n}$$

$$=\frac{1}{2}.$$

We have used here the facts that

1. $\sum_{i=0}^{n}(n \| i)=2^{n}$; and
2. for any n+1 real numbers (or complex numbers, or elements of a commutative ring) $x_0,x_1,\ldots,x_n$ one has

$$2\sum_{i=0}^{n}x_i\sum_{j=i+1}^{n}x_j=\left(\sum_{i=0}^{n}x_i\right)^{2}-\sum_{i=0}^{n}x_i^{2},$$

as one shows by induction on n or by considering $(x_0+x_1+\ldots+x_n)^2$.

Also solved by Gary Jinglin Kuang