Problem for 2002 February

Proposed by Dan Jurca

Does there exist a continuous function $f:\mathbf{R}\to\mathbf{R}$ such that each value attained by f is attained precisely twice?

That is, does there exist a function $f:\mathbf{R}\to\mathbf{R}$ such that

1. f is continuous; and
2. $x \in \mathbf{R} \Rightarrow \exists\,!\, y \in \mathbf{R}$ such that
   i. $y \neq x$, and
   ii. $f(y)=f(x)$?

Solution by the proposer

There does **not** exist such a function. For suppose f is such a function; we derive a contradiction as follows.

First let $f_0:\mathbf{R}\to\mathbf{R}$ by $f_0(x)=f(x)-f(0)$. Then f_0 is continuous and $f_0(0)=0$. Next there exists $y \neq 0$ such that $f(y)=f(0)$; let $f_1:\mathbf{R}\to\mathbf{R}$ by $f_1(x)=f_0(y\cdot x)$. Then f_1 is continuous, and $f_1(1)=f_0(y)=f(y)-f(0)=0=f_0(0) = f_1(0)$. Now $f_1|_{[0,1]}$ is continuous, so attains a minimum and a maximum in [0,1]; let us say a minimum occurs at $u \in [0,1]$ and a maximum occurs at $v \in [0,1]$. Now clearly either $f_1(u) < 0$ or $0 < f_1(v)$, as otherwise f attains the value 0 at more than two points. Now define

$$f_2=\begin{cases} f_1 & \text{if } 0 < f_1(v), \\ -f_1 & \text{if } f_1(v)=0. \end{cases}$$

Then f_2 is continuous, $f_2(0)=0=f_2(1)$, and $f_2|_{[0,1]}$ attains a maximum value different from 0 in the interval [0,1], let us say at the point $a \in [0,1]$. Now there exists a point $b \in \mathbf{R}$ such that $b \neq a$ and $f_2(b)=f_2(a)$. We consider two cases.

First suppose $b \in [0,1]$. Again there are two cases: $b < a$ and $a < b$. We will consider $a < b$, as the other case is similar. Now there exists c such that $a < c < b$, and $0 < f_2(c) < f_2(a)=f_2(b)$. By the intermediate value theorem there exist x_1 between 0 and a such that $f_2(x_1)=f_2(c)$, x_2 between a and c such that $f_2(x_2)=f_2(c)$, x_3 between c and b such that $f_2(x_3)=f_2(c)$, and x_4 between b and 0 such that $f_2(x_4)=f_2(c)$. But it now follows that $f(x_1)=f(x_2)=f(x_3)=f(x_4)$, and of course x_1, x_2,x_3, and x_4 are distinct. Thus f attains some value at least four times, a contradiction.

A similar (and similarly tedious) argument holds if $b \notin [0,1]$.

Hence no such f exists.