Problem for 2002 May

Communicated by Dan Jurca

The following problem in the 2002 April issue of the Canadian mathematics journal *Crux Mathematicorum* is from the Swiss Mathematical Contest, May 17, 1999.

Find all solutions $(x,y,z) \in \mathbf{R}\times\mathbf{R}\times\mathbf{R}$ of the system

$$\frac{4x^2}{1+4x^2}=y, \qquad \frac{4y^2}{1+4y^2}=z, \qquad \frac{4z^2}{1+4z^2}=x.$$

Solution by Dan Jurca

It is clear that (0,0,0) and (1/2,1/2,1/2) are solutions; we show these are the only solutions. Consider $f:\mathbf{R}\to\mathbf{R}$ by

$$f(t) = \frac{4t^2}{1+4t^2}. \quad \text{We find}$$

$$f'(t) = \frac{8t\cdot(1+4t^2)-4t^2\cdot 8t}{(1+4t)^2}$$

$$= \frac{8t}{(1+4t^2)^2},$$

so $t < 0 \Rightarrow f'(t) < 0$, $0 < t \Rightarrow 0 < f'(t)$. Thus

$f(0)=0$, $0 \le f < 1$, $\lim_{\infty} f=1$, f\searrowon $(-\infty,0]$, f\nearrowon $[0,\infty)$.

Now we have x=f(z)=f(f(y))=f(f(f(x))), so that with g=f°f°f, x is a fixed point of g; *i.e.*, g(x)=x. Since f\nearrow on [0,∞), it follows that g\nearrow on [0,∞). Observe that

$$t-f(t)=\frac{t(1-2t)^2}{1+4t^2}=0 \text{ if and only if } t=0 \text{ or } t=1/2,$$

and $0 < t < 1/2$ or $1/2 < t \Rightarrow t < f(t)$. Thus $0 < t < 1/2$ or $1/2 < t \Rightarrow$

t < f(t), so

f(t) < f(f(t)), (since f\nearrow), so

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f(t) < f(f(f(t))), (since f\nearrow), so

t < g(t), (since t < f(t).)

Hence $t \neq 0$ or $t \neq 1/2 \Rightarrow g(t) \neq t$. Thus the only fixed points of g are 0 and 1/2. Therefore x=0 or x=1/2. Now

$$x=0 \Rightarrow y=0 \Rightarrow z=0, \quad \text{and } x=1/2 \Rightarrow y=1/2 \Rightarrow z=1/2.$$

Hence the only solutions are (0,0,0) and (1/2,1/2,1/2), as asserted.