Problem for 2002 June

Communicated by Dan Jurca

This is an old problem, but has not appeared here for a long time.

There are one million dots (points) in the plane; prove that there exists a (straight) line such that exactly half of the dots are on each side of the line.

Solution (from a book of problems by Charles Trigg)

We generalize to the case that there are 2n dots, where n is a positive integer. Let us call this set of points S. Consider the set of all

$$\binom{2n}{2} = n(2n-1)$$

(not necessarily distinct) lines determined by each pair of points chosen from S. These lines do not cover the entire plane-the plane is not the union of finitely many lines. Hence there exists a point, say P, which does not lie on any of the lines, and furthermore lies to the left of some disk the interior of which includes S. Now for each point Q in S the line determined by P and Q contains no point of S other than Q. Therefore as the point Q varies through S we may mark the n-th and the (n+1)-th lines; each line through P between these lines has n points of S on each side.

Also solved by Matthew Hubbard and John Sayer.