Problem for 2002 July

Proposed by Dan Jurca

Suppose n is a positive integer and A_n is the n×n matrix as follows.

$$A_n=\begin{pmatrix} 2 & 1 & 0 & 0 & \dots & 0 \\ 1 & 2 & 1 & 0 & \dots & 0 \\ 0 & 1 & 2 & 1 & \dots & 0 \\ 0 & 0 & 1 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 2 \end{pmatrix}$$

That is, $A_n=(a_{ij})_{1 \le i \le n,\, 1 \le j \le n}$ where

$$a_{ij}=\begin{cases} 2 & \text{if } i=j, \\ 1 & \text{if } |i-j|=1, \\ 0 & \text{if } 2 \le |i-j|. \end{cases}$$

Compute the determinant $\det(A_n)$.

Solution by the proposer

Let $d_n=\det(A_n)$; then we find at once $d_1=2$ and $d_2=3$. If $3 \leq n$, then by expanding along column 1 in A_n we find $d_n=2d_{n-1}-d_{n-2}$. Thus $1 \leq n \Rightarrow d_n=n+1$. For this is clear if n=1 or n=2; if $3 \leq n$, $d_{n-1}=n$, and $d_{n-2}=n-1$, then by the recursion relation we have $d_n=2\times n-(n-1)=n+1$; hence by induction on n we have $1 \leq n \Rightarrow d_n=n+1$, as asserted.

Also solved by Walt Becker, Christopher Doi, John M. Sayer, and Murray Stokely