Problem for 2002 August

Proposed by Dan Jurca

Suppose K is a field, and V is a vector space over K; a *bilinear form* on V is a function

$$b:V\times V\to K$$

which is linear (additive and homogeneous) in each ``slot''; *i.e.*, $\mathbf{u},\mathbf{v},\mathbf{w}\in V$ and $k\in K\Rightarrow$

$$b(\mathbf{u}+\mathbf{v},\mathbf{w}) = b(\mathbf{u},\mathbf{w})+b(\mathbf{v},\mathbf{w}),\text{ and } b(\mathbf{u},\mathbf{v}+\mathbf{w})=b(\mathbf{u},\mathbf{v})+b(\mathbf{u},\mathbf{w});$$
$$\text{and}\quad b(k\mathbf{u},\mathbf{v}) = kb(\mathbf{u},\mathbf{v})=b(\mathbf{u},k\mathbf{v}).$$

One finds immediately that for each bilinear form b on V

$$\mathbf{v}\in V\Rightarrow b(\mathbf{0},\mathbf{v})=0=b(\mathbf{v},\mathbf{0}).$$

The following is a standard exercise in linear algebra.

Suppose V is a finite-dimensional vector space over the field K, and $b:V\times V\to K$ is a bilinear form on V; then the following conditions are equivalent.

a.
   $\{\mathbf{u}\in V\mid b(\mathbf{u},\mathbf{v})=0\ \forall\ \mathbf{v}\in V\}=\{\mathbf{0}\}$;

b.
   $\{\mathbf{v}\in V\mid b(\mathbf{u},\mathbf{v})=0\ \forall\ \mathbf{u}\in V\}=\{\mathbf{0}\}$;

c.
   for each basis $\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_n\}$ of V over K the n×n matrix $B=(b_{ij})_{1\le i\le n,1\le j\le n}$, where $b_{ij}=b(\mathbf{v}_i,\mathbf{v}_j)$, is nonsingular.

(Such a bilinear form is called *non-degenerate*.)

Show that the hypothesis that $\dim_K(V)<\infty$ above is essential. That is, show that if V is an *infinite dimensional* vector space over the field K, then there exists a bilinear form $b:V\times V\to K$ such that

$$\{\mathbf{u}\in V \mid b(\mathbf{u},\mathbf{v})=0\ \forall\ \mathbf{v}\in V\} = \{\mathbf{0}\};$$

$$\text{but}\quad \{\mathbf{v}\in V \mid b(\mathbf{u},\mathbf{v})=0\ \forall\ \mathbf{u}\in V\} \neq \{\mathbf{0}\}.$$

Solution by the proposer

Suppose V is a vector space over the field K, $\dim_K(V)=\infty$, and $\{\mathbf{v}_\alpha \mid \alpha\in A\}$ is a K-basis of V. Thus A is an infinite set and if $\mathbf{v}\in V$, then for each $\alpha\in A$ there exists $k_\alpha\in K$ and there exists a finite subset $S_\mathbf{v}\subset A$ such that

i.
$\sum\{ k_\alpha\mathbf{v}_\alpha \mid \alpha\in A \}=\mathbf{v}$ and

ii.
$\alpha\notin S_\mathbf{v}\Rightarrow k_\alpha=0.$

Now we choose $\alpha_0\in A$ and define $b:V\times V\to K$ as follows.

$$(\alpha,\beta)\in A\times A \Rightarrow b(\mathbf{v}_\alpha,\mathbf{v}_\beta)=\begin{cases} 0 & \text{if } (\beta=\alpha_0) \text{ or } [(\alpha\neq\alpha_0) \text{ and } (\alpha\neq\beta)];\\ 1 & \text{if } (\beta\neq\alpha_0) \text{ and } [(\alpha=\alpha_0) \text{ or } (\alpha=\beta)];\end{cases}$$

and extend by bilinearity.

(One can see, using DeMorgan's laws, that the conditions here are complementary.)

The following sketch shows what the matrix $B=(b_{\alpha\beta})_{(\alpha,\beta)\in A\times A}$ where $b_{\alpha\beta}=b(\mathbf{v}_\alpha,\mathbf{v}_\beta)$ looks like. (Of course without a well-ordering on A the matrix B does not really look like anything.)

$$B=\begin{bmatrix} 1 & & & \dots & 0 & \dots & & & \dots \\ & 1 & & \dots & 0 & \dots & & & \dots \\ & & 1 & \dots & 0 & \dots & & & \dots \\ & & & \ddots & \vdots & & & & \dots \\ 1 & 1 & 1 & \dots & 0 & \dots & 1 & 1 & \dots \end{bmatrix}$$

$$\left[\begin{array}{cccccc} \cdots & \vdots & \ddots & & & \cdots \\ \cdots & 0 & \cdots & 1 & & \cdots \\ \cdots & 0 & \cdots & & 1 & \cdots \\ \cdots & \vdots & \cdots & & & \ddots \\ & & & & & \\ & & & & & \end{array}\right]$$

Here the missing entries are all 0's; the α_0-th column is all 0's; and the α_0-th row is all 1's except that the diagonal entry is 0; and each diagonal entry is 1, except of course the one in the α_0-th row and column.

We claim that if $\mathbf{u} \in V$ and $b(\mathbf{u},\mathbf{v})=0$ for each $\mathbf{v} \in V$, then $\mathbf{u}=\mathbf{0}$. For suppose that $\mathbf{u}=\sum\{k_\alpha\mathbf{v}_\alpha \mid \alpha \in A\}$, where $\alpha \notin S_\mathbf{u} \Rightarrow k_\alpha=0$. If $\alpha_0 \in S_\mathbf{u}$, then choose $\beta \neq \alpha_0$ such that $k_\beta=0$; then $b(\mathbf{u},\mathbf{v}_\beta)=0$, from which $k_{\alpha 0}b(\mathbf{v}_{\alpha 0},\mathbf{v}_\beta)=0$, whence $k_{\alpha 0}=0$. Next, for each $\alpha \in S_\mathbf{u}$, if $\alpha \neq \alpha_0$, since $b(\mathbf{u},\mathbf{v}_\alpha)=0$, then $k_{\alpha 0}b(\mathbf{v}_{\alpha 0},\mathbf{v}_\alpha)+k_\alpha\, b(\mathbf{v}_\alpha,\mathbf{v}_\alpha)=0$, so that $k_\alpha=0$ (since $k_{\alpha 0}=0$ whether $\alpha_0 \in S_\mathbf{u}$ or $\alpha_0 \notin S_\mathbf{u}$). Therefore $\alpha \in A \Rightarrow k_\alpha=0$, and this shows $\mathbf{u}=\mathbf{0}$. However, it is clear that $\mathbf{u} \in V \Rightarrow b(\mathbf{u},\mathbf{v}_{\alpha 0})=0$; and of course $\mathbf{v}_{\alpha 0}$, being an element of a basis, is non-zero.

Thus for this particular bilinear form b, we have

$$\{\mathbf{u} \in V \mid b(\mathbf{u},\mathbf{v}) = 0\ \forall\ \mathbf{v} \in V\}=\{\mathbf{0}\}, \text{ but}$$
$$\mathbf{v}_{\alpha 0} \in \{\mathbf{v} \in V \mid b(\mathbf{u},\mathbf{v}) = 0\ \forall\ \mathbf{u} \in V\} \neq \{\mathbf{0}\},$$

as desired.

No other solution was received.