Problem for 2002 October

Proposed by Dan Jurca

The following is a variation on a problem which, according to the 2002 September issue of the Canadian mathematics journal *Crux Mathematicorum*, appeared in one of the St. Petersburg contests, from 1965 to 1984.

Suppose that you have a calculator which may be used to

i.
add any two real numbers;

ii.
subtract any real number from any real number;

iii.
divide any real number by 24;

iv.
raise any real number to the fourth power.

Show how this calculator may be used to compute the product of any two real numbers.

Solution by Dan Jurca

By repeated addition one can compute nx for each positive integer n and real number x; hence one can compute $x/2=(12x)/24$, $x/3=(8x)/24$, $x/8=(3x)/24$ for each real number x. Thus one can use this calculator to divide by 2, 3, and 8. Next, from $(x+y)^2=x^2+2xy+y^2$ one has $xy=[(x+y)^2-x^2-y^2]/2$, so that if one can compute squares, then one can compute products. Next, from $(x+1)^3=x^3+3x^2+3x+1$ one finds $x^2=[(x+1)^3-x^3-3x-1]/3$, so that if one can compute cubes, then one can compute squares. Next, from

$$(x+1)^4 = x^4+4x^3+6x^2+4x+1 \quad \text{and}$$
$$(x-1)^4 = x^4-4x^3+6x^2-4x+1$$

one finds $x^3=[(x+1)^4-(x-1)^4-8x]/8$, so that if one can compute fourth powers, then one can compute cubes. Hence if one can compute sums and differences, can divide by 24=4!, and can compute fourth powers, then one can compute products.

We remark that from

$$(x+1)^5 = x^5+5x^4+10x^3+10x^2+5x+1 \quad \text{and}$$
$$(x-1)^5 = x^5-5x^4+10x^3-10x^2+5x-1$$

one finds $x^3=[(x+1)^5+(x-1)^5-2x^5-10x]/20$, so that if one can compute sums, differences, fifth powers, and can divide by 5!, then one can compute products.

Does this continue?