# Problem for 2002 November

Communicated by Matthew Hubbard

Find a polynomial p(x,y) such that

$$(x,y) \in \mathbf{R}^2 \Rightarrow 0 < p(x,y)$$

and

$$\inf \{ p(x,y) \mid (x,y) \in \mathbf{R}^2 \} = 0.$$

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Solution by Dan Jurca

Consider the following polynomial, p.

$$\begin{aligned} p(x,y) &= x^2y^2+y^2+2xy+1 \\ &= y^2+(xy+1)^2 \end{aligned}$$

Obviously $\forall(x,y) \in \mathbf{R}^2$: $0 \le p(x,y)$; and clearly $p(x,y)=0$ if and only if both $y=0$ and $xy+1=0$, which is impossible, so that

$$(x,y) \in \mathbf{R}^2 \Rightarrow 0 < p(x,y).$$

Next, if $0 < \varepsilon$, then

$$p\left(\frac{1}{\sqrt{\varepsilon}}, -\sqrt{\varepsilon}\right)=\varepsilon$$

so that im $p=p(\mathbf{R}^2)=\{ t \in \mathbf{R} \mid 0 < t \}$, and therefore

$$\inf \{ p(x,y) \mid (x,y) \in \mathbf{R}^2 \}=0.$$