Problem for 2003 January

Proposed by Dan Jurca

Simplify the following expression.

$$\sqrt[3]{4018030018+\sqrt{16144565225549080325}}+\sqrt[3]{4018030018-\sqrt{16144565225549080325}}$$

Solution by the proposer

Observing that $16144565225549080325=4018030018^2+1$ and writing x for the value of the given expression, we have

$$x=\sqrt[3]{u+\sqrt{u^2+1}}+\sqrt[3]{u-\sqrt{u^2+1}}$$

where u=4018030018. Then from $(a+b)^3=a^3+3a^2b+3ab^2+b^3$, we have

$$x^3=u+\sqrt{u^2+1}+3\sqrt[3]{\left(u+\sqrt{u^2+1}\right)^2\left(u-\sqrt{u^2+1}\right)}+$$

$$+3\sqrt[3]{\left(u+\sqrt{u^2+1}\right)\left(u-\sqrt{u^2+1}\right)^2}+u-\sqrt{u^2+1}$$

$$=2u+3\sqrt[3]{\left((2u^2+1+2u\sqrt{u^2+1})(u-\sqrt{u^2+1})\right)}+$$

$$3\sqrt[3]{\left((u+\sqrt{u^2+1})(2u^2+1-2u\sqrt{u^2+1})\right)}$$

$$=\ldots$$

$$=2u+3\left[\sqrt[3]{\left(-u-\sqrt{u^2+1}\right)}+\sqrt[3]{\left(-u+\sqrt{u^2+1}\right)}\right]$$

$$=2u-3\left[\sqrt[3]{\left(u+\sqrt{u^2+1}\right)}+\sqrt[3]{\left(u-\sqrt{u^2+1}\right)}\right]$$

$$=2u-3x.$$

Therefore the real number x satisfies the equation $x^3+3x-2u=0$. Since the function $f:\mathbf{R}\to\mathbf{R}$ by $f(t)=t^3+3t-2u$ strictly increases on **R** ($0 < 3t^2+3=f'(t)$), the equation has a unique real root. When $u=4018030018$ one finds the unique real root equals 2003; therefore the given expression simplifies to the number 2003.

Also solved by Farid El-Mouchrif and Dangthu Ta