Problem for 2003 March

Proposed by Dan Jurca

Consider the sequence $(t_n)_{n=1}^{\infty}$ of positive integers as follows. $t_1=1$; then we skip the next $1^2=1$ positive integers, and let $t_2=3$; then we skip the next $2^2=4$ positive integers, and let $t_3=8$; then we skip the next $3^2=9$ positive integers, and let $t_4=18$; *etc.* The following sketch shows the first few t_i.

$\underline{1}$ 2 $\underline{3}$ 4 5 6 7 $\underline{8}$ 9 10 11 12 13 14 15 16 17 $\underline{18}$ …

Find a formula for the sum

$$\sum_{i=1}^{n} t_i$$

of the first n terms of this sequence.

Solution by the proposer

We have $t_1=1$, $t_2=3$, $t_3=8$, $t_4=18$, …; precisely we have

$$\begin{aligned} t_1 &= 1, \\ 2 \le i \Rightarrow t_i &= t_{i-1}+(i-1)^2+1. \end{aligned}$$

Eliminating the recursion we find

$$1 \le i \Rightarrow t_i=(2i^3-3i^2+7i)/6.$$

Therefore

$$\sum_{i=1}^{n} t_i = \frac{1}{3}\sum_{i=1}^{n} i^3 - \frac{1}{2}\sum_{i=1}^{n} i^2 + \frac{7}{6}\sum_{i=1}^{n} i$$

$$= \frac{1}{3}\cdot\left[\frac{n(n+1)}{2}\right]^2 - \frac{1}{2}\cdot\frac{n(n+1)(2n+1)}{6} + \frac{7}{6}\cdot\frac{n(n+1)}{2}$$

$$= \frac{(n^2+n)^2}{12} - \frac{(n^2+n)(2n+1)}{12} + \frac{7(n^2+n)}{12}$$

$$= \frac{(n^2+n)(n^2+n-2n-1+7)}{12}$$

$$= \frac{(n^2+n)(n^2-n+6)}{12}$$

$$= \frac{n^4+5n^2+6n}{12}.$$

Also solved by Walt Becker, Christian G. Bowers, Roger W. Doering, Farid El-Mouchrif, Rosta Farzan, Philip Horowitz, Kurt Luoto, Massoud Malek, John Sayer, Murray Stokely, and Winston Teitler

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