

Problem for 2003 April

Communicated by Dan Jurca

The following problem appears in the "Mathematical Mayhem" section of the 2003 March issue of the Canadian mathematics journal *Crux Mathematicorum*.

Five balls numbered 1 to 5 are put into a box. A ball is drawn at random, its number recorded, and the ball is returned to the box. This process is repeated until five numbers have been recorded. If the sum of the numbers is 15, what is the probability that the number 3 was drawn each time?

Solution by Dan Jurca

The probability is clearly $1/N$ where N is the number of ways to write 15 as the sum of five positive integers, $15 = x_1 + x_2 + x_3 + x_4 + x_5$, where $1 \leq i \leq 5 \Rightarrow 1 \leq x_i \leq 5$. To determine this number N we proceed as follows.

More generally, let $N(S, n, m)$ be the number of ways to write $S = x_1 + x_2 + \dots + x_n$ where each x_i is an integer and $1 \leq i \leq n \Rightarrow 1 \leq x_i \leq m$. Then we have

$$N(S, n, m) = \begin{cases} 0 & \text{if } n=1 \text{ and } S < 1 \text{ or } m < S \\ 1 & \text{if } n=1 \text{ and } 1 \leq S \leq m \\ \sum_{i=1}^m N(S-i, n-1, m) & \text{if } 2 \leq n. \end{cases}$$

(For clearly x_1 equals either 1 or 2 or 3 or ... or m .)

Then evaluating $N = N(15, 5, 5)$ we find $N = 381$; hence the desired probability is $1/381$.

The evaluation may be done easily and quickly by filling in the entries of the following table row by row.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	$\frac{1}{5}$
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1	1	1	1	1	0	0	0	0	0	0					
2	0	1	2	3	4	5	4	3	2	1	0	0			
3				6	10	15	18	19	18	15	10	6			
4										68	80	85	80	68	
5															381

Here in each row after row 1 each entry is the sum of the five entries which are in the previous row to the left of the entry being filled.

(The entry in row i and column j here is $N(j,i,5)$; irrelevant entries are left blank.)

Also solved by Richie Hom, P. Horowitz, Yipkei Kwok, Joseph Rios, John Sayer, and Murray Stokely