Problem for 2003 May

Proposed by Dan Jurca

Prove that for each nonnegative integer n

$$\frac{e}{n+2} < \int_0^1 x^n e^x \, dx < \frac{e}{n+1} .$$

Solution by the proposer

Since the exponential function $(x \to e^x)$ is increasing we have $x < 1 \Rightarrow e^x < e^1 = e$; hence $0 \le x \le 1 \Rightarrow x^n e^x \le x^n e$ (equality only if x=0 or x=1); therefore

$$\int_0^1 x^n e^x \, dx < \int_0^1 e x^n \, dx \quad \text{so that}$$

$$\int_0^1 x^n e^x \, dx < e \int_0^1 x^n \, dx = \frac{e}{n+1} .$$

Next consider $\varphi:\mathbf{R}\to\mathbf{R}$ by $\varphi(x)=e^x-ex$. Then $\varphi'(x)=e^x-e$, so $x < 1 \Rightarrow \varphi'(x) < 0$ and $1 < x \Rightarrow 0 < \varphi'(x)$. Thus φ\searrow in $(-\infty,1]$; φ\nearrow in $[1,\infty)$. Hence $\varphi_{min}=\varphi(1)=0$, so that $x \in \mathbf{R} \Rightarrow 0 \le e^x - ex$. That is, $ex \le e^x$, with equality if and only if x=1. Therefore for each nonnegative n we have $0 \le x \Rightarrow ex \cdot x^n \le x^n e^x$, with equality only if x=0 or x=1. It follows that

$$\int_0^1 e x^{n+1} \, dx < \int_0^1 x^n e^x \, dx \quad \text{so}$$

$$\frac{e}{n+2} < \int_0^1 x^n e^x \, dx,$$

and this together with the inequality above yields

$$0 \le n \Rightarrow \frac{e}{n+2} < \int_0^1 x^n e^x \, dx < \frac{e}{n+1},$$

as desired.

Remark. With a little more work one can show

$$\int_0^1 x^n e^x \, dx = e \sum_{i=1}^{\infty} \frac{n!(n+2i-1)}{(n+2i)!}$$

so that the integral can be computed easily and accurately.

Also solved by Farid El Mouchrif, James Farrell, Yipkei Kwok, Kurt Luoto, Massoud Malek, John M. Sayer, and Dangto Ta