Problem for 2003 June

Proposed by Dan Jurca

Consider the Fibonacci sequence

$$(f_n)_{n=0}^{\infty}=(0,1,1,2,3,5,8,13,21,\ldots).$$

That is,

$$f_n=\begin{cases} 0 & \text{if } n=0 \\ 1 & \text{if } n=1 \\ f_{n-2}+f_{n-1} & \text{if } 2 \le n. \end{cases}$$

Then let

$$F(x) = \sum_{n=0}^{\infty} f_n x^n$$

$$=f_0+f_1x+f_2x^2+f_3x^3+f_4x^4+\ldots .$$

a. For which x does the series above converge?

b. Find a formula for the function F(x).

c. Compute F(1/2).

d. Sketch the graph of F.

Solution by the proposer

Let $\varphi = (1+\sqrt{5})/2$ and observe that $-1/\varphi = (1-\sqrt{5})/2$. We recall the following formula.

$$0 \le n \Rightarrow f_n=(\varphi^n-(-1/\varphi)^n)/\sqrt{5}$$

Now obviously $1 < \varphi$, so that by the root test the series above converges if $|x| < 1/\varphi$ and diverges if $1/\varphi < |x|$. Next we observe that (since $0 < \varphi$) $f_n/\varphi^n=1/\sqrt{5}[1-(-1)^n/\varphi^{2n}]$ does not converge to 0 as $n\to\infty$; similarly, $f_n/(-\varphi)^n=1/\sqrt{5}[(-1)^n-1/\varphi^{2n}]$ does not converge to 0 as $n\to\infty$; thus the interval of convergence of the series is precisely $(-1/\varphi,1/\varphi)$. We compute as follows.

$$\begin{aligned} xF(x) &= f_0x+f_1x^2+f_2x^3+f_3x^4+\dots \\ x^2F(x) &= f_0x^2+f_1x^3+f_2x^4+\dots \\ \text{thus}\quad xF(x)+x^2F(x) &= f_0x+f_2x^2+f_3x^3+f_4x^4+\dots \\ &= F(x)-x; \\ \text{hence}\quad F(x) &= \frac{x}{1-x-x^2}\,. \end{aligned}$$

(One might find it amusing to actually carry out this division.)

Therefore $F(1/2)=[(1/2)/(1-1/2-1/4)]=2$.

Also solved by Walt Becker, James Farrell, Massoud Malek, and John Sayer