Problem for 2003 July

Communicated by Dan Jurca

Recall that a function $\varphi:\mathbf{R}\to\mathbf{R}$ is **additive** if

$$\forall x,y \in \mathbf{R}: \quad \varphi(x+y)=\varphi(x)+\varphi(y).$$

The following problem appears on page 36 of *Conjecture and Proof* by Miklós Laczkovich.

Let $f:\mathbf{R}\to\mathbf{R}$ be a function such that

$$f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}$$

for every $x,y \in \mathbf{R}$. Prove that there is a constant c such that $f-c$ is additive.

Solution by Dan Jurca

Let $\varphi:\mathbf{R}\to\mathbf{R}$ by $\varphi(x)=f(x)-f(0)$. We claim that $x \in \mathbf{R} \Rightarrow \varphi(2x)=2\varphi(x)$. For

$$\begin{aligned}
\varphi(x) &= \varphi\left(\frac{2x+0}{2}\right) \\
&= f\left(\frac{2x+0}{2}\right) - f(0) \\
&= \frac{f(2x)+f(0)}{2} - f(0) \\
&= \frac{f(2x)-f(0)+f(0)-f(0)}{}
\end{aligned}$$

$$2$$

$$= \frac{f(2x)-f(0)}{2}$$

$$= \frac{\varphi(2x)}{2}$$

so $\varphi(2x) = 2\varphi(x)$.

But then $x,y \in \mathbf{R} \Rightarrow$

$$\begin{aligned}
\varphi(x+y) &= \varphi\left(\frac{2x+2y}{2}\right) \\
&= f\left(\frac{2x+2y}{2}\right) - f(0) \\
&= \frac{f(2x)+f(2y)}{2} - f(0) \\
&= \frac{f(2x)-f(0)+f(2y)-f(0)}{2} \\
&= \frac{\varphi(2x)+\varphi(2y)}{2} \\
&= \frac{2\varphi(x)+2\varphi(y)}{2} \\
&= \varphi(x)+\varphi(y).
\end{aligned}$$

Therefore with $c=f(0)$ the function $f-c$ is additive.

Also solved by Kirk Demlinger, James Farrell, Kurt Luoto, and Massoud Malek