Problem for 2003 August

Proposed by Dan Jurca

It is well-known (and easy to prove) that if $1 \le n$, then for each pair A, B of $n \times n$ matrices trace(AB)=trace(BA).

Show that if $2 \le n$, then there exist $n \times n$ matrices A, B, C such that trace(ABC) $\ne$ trace(ACB).

Solution by the proposer

Let A, B, and C be $n \times n$ martices as follows.
(Each matrix entry not explicitly shown is a zero.)

$$A=\begin{pmatrix} 1 & 0 & \dots \\ 0 & 0 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \quad B=\begin{pmatrix} 0 & 1 & \dots \\ 0 & 0 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \quad C=\begin{pmatrix} 0 & 0 & \dots \\ 1 & 0 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

Then one has immediately that AB=B, and BC=A, so that ABC=A, whence trace(ABC) = trace(A)=1; however, AC=O, so that ACB=O, whence trace(ACB) = trace(O)=0, so that trace(ABC) $\ne$ trace(ACB).

Also solved by James Farrell, Bolin Hsu, Massoud Malek, Anurag Sethi, and Winston Teitler