Problem for 2003 November

Communicated by Dan Jurca

Prove that there exists no regular pentagon in the plane such that each vertex of the pentagon has integer coordinates.

Solution by Dan Jurca

Suppose P is a polygon such that each coordinate of each vertex is an integer. Then the area of P is either an integer or 1/2 times an integer. Here are two ways to verify this.

1. One can use Green's lemma (or the two-dimensional version of Green's theorem, or Green's theorem in the plane) which asserts that if P(x,y), Q(x,y), and the partial derivatives ∂P/∂y and ∂Q/∂x are continuous functions over the closed region R bounded by the curve C, then

$$\int_C (P\,dx + Q\,dy) = \int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy.$$

Then with P(x,y)=0 and Q(x,y)=x we have

$$\text{area of R} = \int\int_R dx\,dy = \int_C x\,dy.$$

Now if the vertices of P have coordinates (x_1,y_1), (x_2,y_2), …, (x_n,y_n), in order, then with C the curve defined by traversing the edges: (x_1,y_1) to (x_2,y_2) to (x_3,y_3) to … to (x_n,y_n) and back to (x_1,y_1), one finds by evaluating the line integral above that the area A of P is given by

$$A = \frac{1}{2} \left| \sum_{i=1}^{n} (x_i y_{i+1} - x_{i+1} y_i) \right|, \quad \text{where } (x_{n+1},y_{n+1})=(x_1,y_1).$$

Therefore if each x_i is an integer and each y_i is an integer, it follows that 2A is an integer.

2. One can use Pick's theorem: Suppose that P is a polygon each vertex of which is an integer lattice point. Assume there are I lattice points in the interior of P and B lattice points on the boundary of P. Then the area of P equals I+B/2−1. (See `http://www.cut-the-knot.org/ctk/Pick_proof.shtml`). Thus twice this area is an integer.

Therefore if P is a polygon each vertex of which has integer coordinates, then the area of P is a rational number.

Next one finds that the area of a regular n-gon with side s is given by

$$A = \frac{ns^2}{4} \cot \pi/n.$$

Now if a regular n-gon has vertices with rational coordinates, then clearly the length of a side, s, is the square root of a rational number, so that s^2 is a rational number, and the area is a rational multiple of $\cot\pi/n$. In the case n=5 we have

$$\cot \frac{\pi}{5} = \frac{\sqrt{25+10\sqrt{5}}}{5},$$

which is irrational (else its square is rational, so that $\sqrt{5}$ is rational). Therefore if P is a regular pentagon and the length of a side of P is rational, then the area of P is irrational.

It follows that no regular pentagon has vertices with rational coordinates; in particular, no regular pentagon has vertices with integer coordinates.

Also solved by Kurt Luoto