Problem for 2003 December

Communicated by Dan Jurca

Suppose that k is a real number, $1 < k$, and $f:\mathbb{R}^n \to \mathbb{R}^n$ is a surjection with the following property.

$$x \in \mathbb{R}^n \text{ and } y \in \mathbb{R}^n \Rightarrow k\cdot\|x-y\| \le \|f(x)-f(y)\|$$

Show that there exists a unique fixed point of f.

Solution by Dan Jurca

First we show that f is injective. For if $f(x)=f(y)$, then $k\cdot\|x-y\| \le \|f(x)-f(y)\|=\|0\|=0$, so that $\|x-y\|=0$, whence $x=y$. Now since f is injective and surjective, it follows that f is bijective; hence there exists a unique inverse, say $g:\mathbb{R}^n\to\mathbb{R}^n$, of f. We next show that g is a contraction. For if $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$, then if $g(x)=u$ and $g(y)=v$, then $f(u)=x$ and $f(v)=y$; hence $k\cdot\|g(x)-g(y)\|=k\cdot\|u-v\| \le \|f(u)-f(v)\|=\|x-y\|$, so that $\|g(x)-g(y)\| \le 1/k\cdot\|x-y\|$, and $1/k$ is a constant less than 1. Hence g is a contraction. Since $\mathbb{R}^n$ is a complete metric space, it follows from the Banach contraction theorem that there exists a (unique) fixed point, say p, of g. But if $g(p)=p$, then $f(g(p))=f(p)$, so that (since $f\circ g=id_{\mathbb{R}^n}$) $f(p)=p$. Thus there exists a fixed point of f. Uniqueness follows at once, since if $f(p_1)=p_1$ and $f(p_2)=p_2$, then $k\cdot\|p_1-p_2\| \le \|f(p_1)-f(p_2)\|=\|p_1-p_2\|$ so $(k-1)\|p_1-p_2\|=0$; since $k \ne 1$, it follows that $\|p_1-p_2\|=0$, so that, finally, $p_1=p_2$.

Thus there exists a unique fixed point of f.

Also solved by Massoud Malek and John Sayer