Problem for 2004 January

Proposed by Matthew Hubbard

There are three non-abelian groups of order 12:

i.
the symmetries of the tetrahedron, A_4;

ii.
the dihedral group of the hexagon, D_6; and

iii.
a group known as T or as Q_6, which has generators a and b with relations

$$a^6=1,\ b^2=a^3=(ab)^2.$$

Since each of these is of order 12, each can be embedded in the symmetric group S_{12}.
It is well-known that A_4 can be embedded in S_4 and D_6 can be embedded in S_6.

What is the least n such that Q_6 can be embedded in S_n?

Solution by the proposer

Since a is a permutation of order six it must be a disjoint union of 6-cycles, 3-cycles, and 2-cycles; if there is no 6-cycle, there must be at least one 3-cycle and one 2-cycle, and a^3 is made up of disjoint 2-cycles. Since b is of order four it is made up of disjoint 4-cycles and 2-cycles, and b^2 is made up of disjoint 2-cycles generated by the 4-cycles of b; likewise, ab is of order four, and $(ab)^2$ is made up of disjoint 2-cycles. The smallest number of elements we can use to create a, b, and ab meeting our criteria is seven, and an example follows.

a =(123)(46)(57), so a^3=(46)(57)

b =(12)(4567), so b^2=(46)(57)

ab =(13)(4765), so $(ab)^2$=(46)(57)

All the generating relations are satisfied, so a and b generate Q_6.