Problem for 2004 April

Communicated by Dan Jurca

According to the 2004 March issue of the Canadian mathematics journal *Crux Mathematicorum* the following problem appeared in the 2003 Croatian Mathematical Society National Competition, Junior Level.

Prove that if the product of the positive real numbers x, y, and z is equal to 1, and

$$x+y+z \le \frac{1}{x} + \frac{1}{y} + \frac{1}{z},$$

then for each nonnegative integer n

$$x^n+y^n+z^n \le \frac{1}{x^n} + \frac{1}{y^n} + \frac{1}{z^n}.$$

Solution by Dan Jurca

Assume $0 < x \le y \le z$, so that $x \le 1$, else $1 < xyz$. Since $xyz=1$, we have $z=1/(xy)$; next from

$$\begin{aligned} x+y+\frac{1}{xy} &\le \frac{1}{x} + \frac{1}{y} + xy \quad \text{we have} \\ x^2y+xy^2+1 &\le y+x+x^2y^2, \quad \text{so} \\ 0 &\le (x^2-x)y^2-(x^2-1)y+(x-1) \quad \text{whence} \\ 0 &\le (x-1)[xy^2-(x+1)y+1]. \end{aligned}$$

Now if $x=1$, then $yz=1$ so $z=1/y$ and the inequality to be shown becomes simply $y^n+(1/y)^n \le 1/(y^n)+y^n$, which is obvious; hence assume $x < 1$. Now $xy^2-(x+1)y+1=0$ iff $y=1$ or $y=1/x$, so since $xy^2-(x+1)y+1 \le 0$, we deduce that $1 \le y \le 1/x$.

Now suppose α is a positive real number, let $S=\{(u,v) \in \mathbf{R}^2 \mid 0 < u \le 1 \text{ and } 1 \le v \le 1/u\}$, and consider $f:S\to\mathbf{R}$ by $f(u,v)=u^\alpha+v^\alpha+1/(uv)^\alpha-1/u^\alpha-1/v^\alpha-(uv)^\alpha$. We observe that $(x,y) \in S$ and

$$
\begin{aligned}
f(u,1) &= 0,\\
f\left(u,\frac{1}{u}\right) &= 0; \quad \text{and}\\
1 < v \Rightarrow \lim_{u\to 0+} f(u,v) &= -\infty, \quad \text{and}\\
\frac{\partial f}{\partial u}(u,v) &= \alpha u^{\alpha-1}-\alpha\frac{1}{u^{\alpha+1}v^\alpha}+\alpha\frac{1}{u^{\alpha+1}}-\alpha u^{\alpha-1}v^\alpha\\
&= \frac{\alpha(1-v^\alpha)}{u^{\alpha+1}v^\alpha}[(uv)^\alpha u^\alpha-1],
\end{aligned}
$$

and since $1-v^\alpha \le 0$ and $(uv)^\alpha u^\alpha-1 \le 0$, it follows that $0 \le \partial f/\partial u$, so that for each v_0, $1 < v_0$, the function $g:(0,1/v_0]\to\mathbf{R}$ by $g(u)=f(u,v_0)$ increases from $-\infty$ at 0+ to 0 at $1/v_0$, so we have $f(u,v) \le 0$ for each $(u,v) \in S$. Hence (with $u=x$ and $v=y$)

$$
x^\alpha+y^\alpha+\frac{1}{(xy)^\alpha} \le \frac{1}{x^\alpha}+\frac{1}{y^\alpha}+(xy)^\alpha,
$$

and the desired inequality holds for each nonnegative real number n.