Problem for 2004 May

Communicated by Matthew Hubbard

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There are 101 points (ants) on the interval [0,1], and one of them, call it p^*, is at 1/2. Each point has its own randomly selected starting direction, either left or right, and each point moves at the rate of one unit per minute. When two points collide they bounce off in opposite directions without losing speed; and when a point collides with an endpoint of the interval, 0 or 1, it bounces back without losing speed.

What is the probability that after one minute p^* is back at the point 1/2?

Solution by Kurt Luoto

Imagine that on the head of each ant there sits a dust mite. The dust mite rides on the ant's head until the ant collides with another ant, at which time the dust mite jumps onto the head of the other ant (i.e. the dust mites on the two colliding ants trade places). If over time we track the position of mite $\boldsymbol{m_i}$ ($1 \leq i \leq 101$) which starts out on the head of ant $\boldsymbol{p_i}$ at position $\boldsymbol{a_i}$, we see that the mite initially travels in the same direction as the ant $\boldsymbol{p_i}$ and continues in that direction as it hops from ant to ant until at some point the ant it is traveling on collides with one of the endpoints 0 or 1 at which point the mite's direction changes. Note that the mites travel at the same speed as the ants (one unit per minute). For example, after two minutes the mite $\boldsymbol{m_i}$ is back at its original position, $\boldsymbol{a_i}$ and therefore there must be an ant at each of the original positions, and since the order of the ants on the interval never changes, this means that each ant is back at its original position after two minutes.

Now after one minute, the mite $\boldsymbol{m_i}$ is at position ($1 - \boldsymbol{a_i}$), so there must be some ant at position ($1 - \boldsymbol{a_i}$) after one minute. In other words, the distribution of the ants on [0,1] at one minute is the mirror image, reflected around the point ½, of their distribution at the starting time. Since $\boldsymbol{p^*}$ started at position ½, that means that there is *some* ant at position ½ after one minute, though this will be $\boldsymbol{p^*}$ if and only if $\boldsymbol{p^*}$ is the middle ant, i.e. the 51st ant counting from the left. Thus the probability that $\boldsymbol{p^*}$ is back at its starting position after one minute is exactly the probability that $\boldsymbol{p^*}$ is the middle ant.

The problem statement does not say how initial positions are determined, but if we assume that the initial positions of the 100 ants other than $\boldsymbol{p}^*$ are selected from [0,1] independently from each other with uniform probability across the interval, then this is the probability that exactly 50 of the ants initially start out in the interval [0, ½).
This probability is , which is approximately 0.0795892.