Problem for 2004 July

Proposed by Dan Jurca

Suppose that $x_i \in \mathbf{R}$ for i=1,2,…,n, and $1 \le x_i$ for each i.

Prove the following.

$$3^n \prod_{i=1}^{n} (x_i^2 - x_i) \le \prod_{i=1}^{n} (x_i^3 - 1)$$

Solution by the proposer

If t is a nonzero real number, then

$$\begin{aligned}
t + \frac{1}{t} &= \frac{t^2+1}{t} \\
&= \frac{2t + t^2 - 2t + 1}{t} \\
&= 2 + \frac{(t-1)^2}{t}, \quad \text{so that if } 0 < t, \text{ then} \\
2 &\le t + \frac{1}{t}. \quad \text{Hence, for each i} \\
3 &\le x_i + \frac{1}{x_i} + 1 \quad \text{so (since } 0 < x_i) \\
3x_i &\le x_i^2 + 1 + x_i, \quad \text{whence (since } 0 \le x_i - 1) \\
3x_i(x_i - 1) &\le (x_i - 1)(x_i^2 + x_i + 1) \quad \text{so that} \\
3(x_i^2 - x_i) &\le x_i^3 - 1,
\end{aligned}$$

from which, by taking the product of all n factors, the desired inequality follows.

Also solved by Dangthu Ta