Problem for 2004 Aug

Communicated by Dan Jurca

Let P, Q, R, and S be four distinct points on a circle in clockwise order, and let A, B, C, and D be the midpoints of the arcs PQ, QR, RS, and SP, respectively. Prove that the line segments AC and BD are perpendicular.

Diagram

Solution by Massoud Malek

Let X be the point of intersection of line segments AC and BD; then

$$
\begin{aligned}
\angle AXB &= \angle ACB + \angle CBD \\
&= \pi - (\angle CAB + \angle ABD), \qquad \text{so that} \\
2\angle AXB &= \pi + \angle ACB + \angle CBD - \angle CAB - \angle ABD \\
&= \pi + \frac{\widehat{AQ} + \widehat{QB}}{2} + \frac{\widehat{CS} + \widehat{SD}}{2} - \frac{\widehat{BR} + \widehat{RC}}{2} - \frac{\widehat{DP} + \widehat{PA}}{2} \\
&= \pi + \frac{\widehat{AQ} - \widehat{PA}}{2} + \frac{\widehat{QB} - \widehat{BR}}{2} + \frac{\widehat{CS} - \widehat{RC}}{2} + \frac{\widehat{SD} - \widehat{DP}}{2} \\
&= \pi + \frac{0}{2} + \frac{0}{2} + \frac{0}{2} + \frac{0}{2} \\
&= \pi,
\end{aligned}
$$

whence $\angle AXB = \pi/2$, and line segments AC and BD are perpendicular.

Also solved by Karen Nelson, John M. Sayer, and Sizhuo Shi