Problem for 2004 November

Communicated by Dan Jurca

Show that

$$\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+5\sqrt{1+\dots}}}}} = 3.$$

Remark. This problem probably originated with Ramanujan. For some interesting background material see Robert Kanigel, *The Man Who Knew Infinity*, page 86.

Solution by Dan Jurca

$$3 = \sqrt{9}$$

$$= \sqrt{1+8}$$

$$= \sqrt{1+2\cdot 4}$$

$$= \sqrt{1+2\sqrt{16}}$$

$$= \sqrt{1+2\sqrt{1+15}}$$

$$= \sqrt{(1+2\sqrt{1+3\cdot 5}}$$

$$= \sqrt{(1+2\sqrt{(1+3\sqrt{25}}}$$

$$= \sqrt{(1+2\sqrt{(1+3\sqrt{1+24}}}$$

$$= \sqrt{(1+2\sqrt{(1+3\sqrt{1+4\cdot 6}}}$$

$$= \sqrt{(1+2\sqrt{(1+3\sqrt{1+4\surd\{36\}}}}$$

$$= \sqrt{(1+2\sqrt{(1+3\sqrt{1+4\surd\{1+35\}}}}$$

$$= \sqrt{(1+2\sqrt{(1+3\sqrt{1+4\surd\{1+5\cdot 7\}}}}$$

$$= \sqrt{(1+2\sqrt{(1+3\sqrt{1+4\surd\{1+5\surd\{49\}\}}}}$$

$$= \sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+5\sqrt{1+48}}}}}$$

$$= \sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+5\sqrt{1+6\cdot 8}}}}}$$

$$= \sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+5\sqrt{1+\ldots}}}}}$$