Problem for 2005 January

Proposed by Dan Jurca

By a *power* we mean a real number p of the form $p=a^b$ where

- a is a positive integer;
- b is an integer; and
- $2 \le b$.

For each real number x let P(x) equal the number of powers p such that $p \le x$. Examples: P(10)=4 and P(100)=13. Prove

$$P(x) \sim \sqrt{x}; \text{ i.e.,}$$

$$\lim_{x\to\infty} \frac{P(x)}{\sqrt{x}} = 1.$$

Solution by the proposer

Suppose x is a real number and $1 \le x$.

Then $1 \le a \le \lfloor\sqrt{x}\rfloor \Rightarrow a^2 \le x$, so that $\lfloor\sqrt{x}\rfloor \le P(x)$.
Next, if $2 \le a$ and $a^b \le x$, then $b\log_2 a \le \log_2 x$, so, since $1 \le \log_2 a$, we must have $b \le \log_2 x/\log_2 a \le \log_2 x$. It follows that there is no b-th power (other than 1^b) less than or equal to x if $\lfloor\log_2 x\rfloor < b$. Also, $a^b \le x \Rightarrow a \le x^{1/b}$. Therefore the number of b-th powers less than or equal to x is bounded by $x^{1/b}$. Hence

$$\begin{aligned} P(x) &\le x^{1/2}+x^{1/3}+x^{1/4}+\ldots+x^{1/\lfloor\log_2 x\rfloor} \\ &=\sqrt{x}+(x^{1/3}+x^{1/4}+\ldots+x^{1/\lfloor\log_2 x\rfloor}) \\ &\le \sqrt{x}+(x^{1/3}+x^{1/3}+\ldots+x^{1/3}) \end{aligned}$$

$$=\sqrt{x}+(\lfloor \log_2 x\rfloor-2)\cdot x^{1/3}$$
$$\le \sqrt{x}+\log_2 x\cdot x^{1/3}.$$

Therefore

$$\lfloor\sqrt{x}\rfloor \le P(x) \le \sqrt{x}+\log_2 x\cdot x^{1/3}, \quad \text{whence}$$
$$\frac{\lfloor\sqrt{x}\rfloor}{\sqrt{x}} \le \frac{P(x)}{\sqrt{x}} \le 1+\frac{\log_2 x\cdot x^{1/3}}{\sqrt{x}}, \quad \text{so}$$
$$\frac{\lfloor\sqrt{x}\rfloor}{\sqrt{x}} \le \frac{P(x)}{\sqrt{x}} \le 1+\frac{\log_2 x}{x^{1/6}}.$$

From the ``squeeze theorem'' and l'Hôpital's rule it follows that

$$\lim_{x\to\infty} \frac{P(x)}{\sqrt{x}} =1.$$

Remark. A more detailed analysis shows

$$1 \le x \Rightarrow p(x)=1-\sum_{r=2}^{\lfloor \log_2 x\rfloor} \mu(r)(\lfloor x^{1/r}\rfloor-1)$$

where μ is the Möbius function.
