Problem for 2005 March

Communicated by Dan Jurca

Find the value of the following limit.

$$\lim_{n\to\infty}\left[\frac{n}{(n+1)^2}+\frac{n}{(n+2)^2}+\frac{n}{(n+3)^2}+\ldots+\frac{n}{(2n)^2}\right]$$

Solution by Dan Jurca

The function $f:(0,\infty)\to\mathbf{R}$ by $f(x)=1/x^2$ strictly decreases, so that

$$2 \le k \Rightarrow \int_k^{k+1}\frac{1}{x^2}\,dx < \frac{1}{k^2} < \int_{k-1}^{k}\frac{1}{x^2}\,dx;$$

hence by summing we find

$$1 \le n \Rightarrow \int_{n+1}^{2n+1}\frac{1}{x^2}\,dx < \frac{1}{(n+1)^2}+\ldots+\frac{1}{(2n)^2} < \int_{n}^{2n}\frac{1}{x^2}\,dx \quad \text{so that}$$

$$\frac{1}{n+1}-\frac{1}{2n+1} < \frac{1}{(n+1)^2}+\ldots+\frac{1}{(2n)^2} < \frac{1}{n}-\frac{1}{2n} \quad \text{from which}$$

$$\frac{n}{2n^2+3n+1} < \frac{1}{(n+1)^2}+\ldots+\frac{1}{(2n)^2} < \frac{1}{2n} \text{ ;hence}$$

$$1 \le n \Rightarrow \frac{n^2}{2n^2+3n+1} < \frac{n}{(n+1)^2}+\ldots+\frac{n}{(2n)^2} < \frac{1}{2},$$

so that by the squeeze theorem

$$\lim_{n\to\infty} \left[\frac{n}{(n+1)^2} + \frac{n}{(n+2)^2} + \frac{n}{(n+3)^2} + \ldots + \frac{n}{(2n)^2} \right] = \frac{1}{2} .$$

Also solved by:

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