Problem for 2005 May

Proposed by Dan Jurca

Suppose $f:[-1,1)\to\mathbf{R}$ as follows.

$$f(x)=\sqrt{\left(\frac{1+x}{1-x}\right)}$$

Write the Maclaurin series for f(x).

Solution by the proposer

We have

$$f(x) = 1\cdot\sqrt{\left(\frac{1+x}{1-x}\right)}$$

$$=\sqrt{\left(\frac{1+x}{1+x}\right)}\cdot\sqrt{\left(\frac{1+x}{1-x}\right)}$$

$$=(1+x)\cdot\frac{1}{\sqrt{1-x^2}}$$

$$=(1+x)\cdot\frac{1}{\sqrt{1-u}},$$

if $-1 < x < 1$ and $u=x^2$. Now with $\varphi(u)=(1-u)^{-1/2}$, we have, by an easy induction on n, that

$$0 \le n \Rightarrow \varphi^{(n)}(u) = \frac{(2n)!}{2^{2n}n!}(1-u)^{-(2n+1)/2}, \quad \text{so that}$$

$$0 \le n \Rightarrow \varphi^{(n)}(0) = \frac{(2n)!}{2^{2n}n!}.$$

Therefore

$$\frac{1}{\sqrt{1-u}} = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}n!n!}u^n, \quad \text{so that}$$

$$-1 < x < 1 \Rightarrow \frac{1}{\sqrt{1-x^2}} = \sum_{n=0}^{\infty} \frac{1}{4^n}\binom{2n}{n}x^{2n}, \quad \text{whence}$$

$$-1 < x < 1 \Rightarrow f(x) = (1+x)\sum_{n=0}^{\infty} \frac{1}{4^n}\binom{2n}{n}x^{2n}$$

$$= \sum_{n=0}^{\infty} a_n x^n \quad \text{where}$$

$$a_n = \frac{1}{4^{\lfloor n/2\rfloor}}\binom{2\lfloor n/2\rfloor}{\lfloor n/2\rfloor}.$$

Since

$$\lim_{k\to\infty} \frac{1}{4^k} \binom{2k}{k} = 0,$$

the series above is valid for $x=-1$ also.

Also solved by Massoud Malek