Problem for 2005 June

Communicated by Dan Jurca

On page 15 of *An Introduction to Modern Cosmology* by Andrew Liddle appears the following integral.

$$\int_0^\infty \frac{y^3\,dy}{e^y - 1}$$

The author asserts, ``The integral is not particularly easy to compute, but you might like to try it as a challenge. The answer is $\pi^4/15$,… ". Thus, the problem becomes the following: Prove

$$\int_0^\infty \frac{y^3\,dy}{e^y - 1} = \frac{\pi^4}{15}.$$

Solution

The following is taken from pages 59 and 60 of *Gamma* by Julian Havil.

We have

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\,dt \quad \text{for } x > 0$$

and make the change of variable t=ru to get

$$\Gamma(x) = \int^\infty (ru)^{x-1} e^{-ru} r\,du = r^x \int^\infty u^{x-1} e^{-ru}\,du.$$

0 0

Hence

$$\frac{1}{r^x} = \frac{1}{\Gamma(x)} \int_0^\infty u^{x-1} e^{-ru}\, du$$

and

$$\zeta(x) = \sum_{r=1}^{\infty} \frac{1}{r^x} = \frac{1}{\Gamma(x)} \sum_{r=1}^{\infty} \int_0^\infty u^{x-1} e^{-ru}\, du$$

$$= \frac{1}{\Gamma(x)} \int_0^\infty u^{x-1} \sum_{r=1}^{\infty} e^{-ru}\, du,$$

having pushed the sigma through the integral, and summing the infinite geometric series results in

$$\zeta(x) = \frac{1}{\Gamma(x)} \int_0^\infty u^{x-1} \frac{e^{-u}}{1-e^{-u}}\, du$$

so we get ``a beautiful formula".

$$\zeta(x)\Gamma(x) = \int_0^\infty \frac{u^{x-1}}{e^u-1}\, du$$

which is valid for $x \notin \{\ldots,-2,-1,0,1\}$; … .

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Since $\zeta(4)=\pi^4/90$ and $\Gamma(4)=3!=6$ we therefore have (with $x=4$)

$$\int_0^\infty \frac{y^3\,dy}{e^y-1} = \frac{\pi^4}{15}\,.$$

Also solved by Jan van Delden and Massoud Malek