Problem for 2005 July

Communicated by Dan Jurca

Suppose that a, b, and c are real numbers; evaluate the following limit.

$$\lim_{t\to 0} \ln\left(\frac{1}{t}\int_0^t (1+a\sin bx)^{c/x}\,dx\right)$$

Solution by Dan Jurca

Using continuity of the natural logarithm function, l Hôpital's rule, and the fundamental theorem of calculus we find

$$\lim_{t\to 0} \ln\left(\frac{1}{t}\int_0^t (1+a\sin bx)^{c/x}\,dx\right) = \ln \lim_{t\to 0} \frac{\int_0^t (1+a\sin bx)^{c/x}\,dx}{t}$$

$$=\ln \lim_{t\to 0} \frac{\frac{d}{dt}\int_0^t (1+a\sin bx)^{c/x}\,dx}{\frac{d}{dt}t}$$

$$=\ln \lim_{t\to 0} \frac{(1+a\sin bt)^{c/t}}{1}$$

$$= \lim_{t\to 0} \ln((1+a\sin bt)^{c/t})$$

$$= \lim_{t\to 0} \frac{c}{t}\ln(1+a\sin bt)$$

$$=c \lim \frac{\ln(1+a\sin bt)}{}$$

$$\lim_{t\to 0} \frac{\ }{t}$$

$$= c \lim_{t\to 0} \frac{\frac{d}{dt}\ln(1+a\sin bt)}{\frac{d}{dt}t}$$

$$= c \lim_{t\to 0} \frac{\frac{1}{1+a\sin bt}\cdot ab\cos bt}{1}$$

$$= c \lim_{t\to 0} \frac{ab\cos bt}{1+a\sin bt}$$

$$= c\cdot ab$$

$$= abc.$$

Also solved by John M. Sayer