Problem for 2005 September

Proposed by Dan Jurca

Diagram

The rectangle above has width a and length b. Find the area of the rhombus with vertices at the centers of the circles inscribed in the four triangles formed by the diagonals of the rectangle.

Solution by the proposer

Consider an isosceles triangle with base b, altitude h, and inscribed circle of radius r. Equating the area of the triangle with the sum of the areas of the three smaller traingles shows

$$\frac{1}{2}bh = \frac{1}{2}b\times r + 2\left(\frac{1}{2}\sqrt{\left(\frac{b}{2}\right)^2 + h^2}\times r\right), \quad \text{so}$$

$$r = \frac{bh}{b+\sqrt{b^2+4h^2}}.$$

Hence the radii r_1 of the larger circle shown in the rectangle and r_2 of the smaller circle are given by

$$r_1 = \frac{a\cdot b/2}{a+\sqrt{a^2+4(b/2)^2}}$$

$$= \frac{ab/2}{a+\sqrt{a^2+b^2}}$$

$$= \frac{ab/2}{a+c}, \qquad \text{where } c=\sqrt{a^2+b^2}; \quad \text{and similarly}$$

$$r_2 = \frac{ab/2}{b+c}.$$

Thus the diagonals of the rhombus are

and

$$b-2r_1 = b-\frac{ab}{a+c}$$

$$= \frac{bc}{a+c}$$

$$a-2r_2 = a-\frac{ab}{b+c}$$

$$= \frac{ac}{b+c},$$

whence the area of the rhombus equals

$$\frac{1}{2} \times \frac{bc}{a+c} \times \frac{ac}{b+c} = \frac{1}{2} \frac{abc^2}{(a+c)(b+c)}.$$

Also solved by Kelly Hubble, Bill Nico, and John M. Sayer